Verification

Lecture 15

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REVIEW: Bisimulation equivalence

Let $TS_i = (S_i, Act_i, \rightarrow_i, I_i, AP, L_i)$, i=1, 2, be transition systems
A bisimulation for (TS_1, TS_2) is a binary relation $\mathcal{R} \subseteq S_1 \times S_2$ such that:

1. $\forall s_1 \in I_1 \, \exists s_2 \in I_2. \, (s_1, s_2) \in \mathcal{R}$ and $\forall s_2 \in I_2 \, \exists s_1 \in I_1. \, (s_1, s_2) \in \mathcal{R}$
2. for all states $s_1 \in S_1$, $s_2 \in S_2$ with $(s_1, s_2) \in \mathcal{R}$ it holds:

   2.1 $L_1(s_1) = L_2(s_2)$

   2.2 if $s_1' \in Post(s_1)$ then there exists $s_2' \in Post(s_2)$ with $(s_1', s_2') \in \mathcal{R}$

   2.3 if $s_2' \in Post(s_2)$ then there exists $s_1' \in Post(s_1)$ with $(s_1', s_2') \in \mathcal{R}$

TS_1 and TS_2 are bisimilar, denoted $TS_1 \sim TS_2$, if there exists a bisimulation for (TS_1, TS_2)

REVIEW: Bisimulation on states

$\mathcal{R} \subseteq S \times S$ is a bisimulation on TS if for any $(q_1, q_2) \in \mathcal{R}$:

- $L(q_1) = L(q_2)$
- if $q_1' \in Post(q_1)$ then there exists an $q_2' \in Post(q_2)$ with $(q_1', q_2') \in \mathcal{R}$
- if $q_2' \in Post(q_2)$ then there exists an $q_1' \in Post(q_1)$ with $(q_1', q_2') \in \mathcal{R}$

q_1 and q_2 are bisimilar, $q_1 \sim_{TS} q_2$, if $(q_1, q_2) \in \mathcal{R}$ for some bisimulation $\mathcal{R}$ for TS

$q_1 \sim_{TS} q_2$ if and only if $TS_{q_1} \sim TS_{q_2}$

REVIEW: CTL* equivalence

States q_1 and q_2 in *TS* (over *AP*) are CTL*-equivalent:

$$q_1 \equiv_{CTL^*} q_2 \quad \text{if and only if} \quad (q_1 \models \Phi \text{ iff } q_2 \models \Phi)$$

for all CTL* state formulas over *AP*

$$TS_1 \equiv_{CTL^*} TS_2 \quad \text{if and only if} \quad (TS_1 \models \Phi \text{ iff } TS_2 \models \Phi)$$

<u>for any sublogic of CTL*, logical equivalence is defined analogously</u>

Bisimulation vs. CTL* and CTL equivalence

Let TS be a finite transition system and s, s' states in TS

The following statements are equivalent:

(1) $s \sim_{TS} s'$

(2) s and s' are CTL-equivalent, i.e., $s \equiv_{CTL} s'$

(3) s and s' are CTL*-equivalent, i.e., $s \equiv_{CTL^*} s'$

this is proven in three steps: $\equiv_{CTL} \;\subseteq\; \sim \;\subseteq\; \equiv_{CTL^*} \;\subseteq\; \equiv_{CTL}$

important: equivalence is also obtained for any sub-logic containing $\neg$, $\wedge$ and X

REVIEW: The importance of this result

- CTL and CTL* equivalence coincide
  - despite the fact that CTL* is more expressive than CTL
- Bisimilar transition systems preserve the same CTL* formulas
  - and thus the same LTL formulas (and LT properties)
- Non-bisimilarity can be shown by a single CTL (or CTL*) formula
  - $TS_1 \models \Phi$ and $TS_2 \not\models \Phi$ implies $TS_1 \not\sim TS_2$
- You even do not need to use an until-operator!
- To check $TS \models \Phi$, it suffices to check $TS/\sim \models \Phi$

Computing bisimulation quotients

A partition $\Pi = \{B_1, \ldots, B_k\}$ of S is a set of nonempty ($B_i \neq \emptyset$) and pairwise disjoint blocks B_i that decompose S ($S = \biguplus_{i=1,\ldots k} B_i$).
A partition defines an equivalence relation $\sim$
$((q, q') \in \sim \Leftrightarrow \exists B_i \in \Pi.\, q, q' \in B_i)$.
Likewise, an equivalence relation $\sim$ defines a partition $\Pi = S/\sim$.
A nonempty union $C = \biguplus_{i \in I} B_i$ of blocks is called a superblock.

A block B_i of a partition Π is called stable w.r.t. a set B if either
$B_i \cap Pre(B) = \emptyset$, or $B_i \subseteq Pre(B)$.

$(Pre(B) = \{q \in S \mid Post(q) \cap B \neq \emptyset\})$

A partition Π is called stable w.r.t. a set B if all blocks of Π are.

Lemma 1. A partition Π with consistently labeled blocks is stable with respect to all of its (super)blocks if, and only if, it defines a bisimulation relation.

Partition refinement

For two partitions $\Pi = \{B_1, \ldots, B_k\}$ and $\Pi' = \{B'_1, \ldots, B'_j\}$ of S, we say that Π is finer than Π' iff every block of Π' is a superblock of Π.

For a given partition $\Pi = \{B_1, \ldots, B_k\}$, we call a (super)block C of Π a splitter of a block B_i / the partition Π if B_i / Π is not stable w.r.t. C.

$\text{Refine}(B_i, C)$ denotes $\{B_i\}$ if B_i is stable w.r.t. C, and $\{B_i \cap Pre(C), B_i \setminus Pre(C)\}$ if C is a splitter of C.

$\text{Refine}(\Pi, C) = \biguplus_{i=1,\ldots,k} \text{Refine}(B_i, C)$.

Lemma 2. $\text{Refine}(\Pi, C)$ is finer than Π.

An algorithm for bisimulation quotienting

Input: Transition system $(S, Act, \rightarrow, I, AP, L)$
Output: Bisimulation quotient

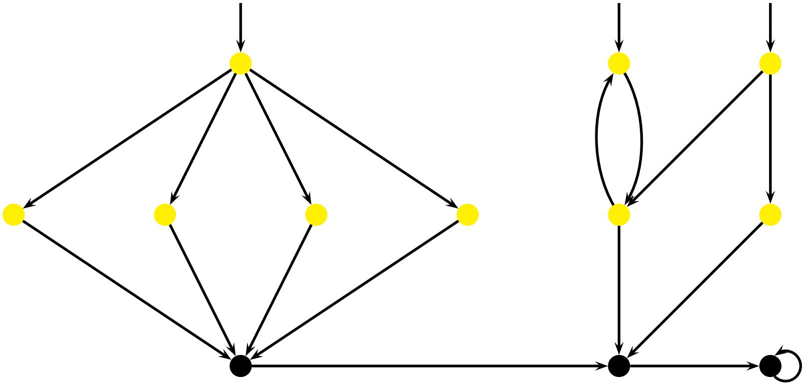
1. $\Pi = S/{\sim_{AP}}$ $\qquad (q, q') \in {\sim_{AP}} \Leftrightarrow L(q) = L(q')$
2. while some block $B \in \Pi$ is a splitter of Π — loop invariant: Π is coarser than $S/{\sim_{TS}}$
   - 2.1 pick a block B that is a splitter of Π
   - 2.2 $\Pi = \text{Refine}(\Pi, B)$
3. return Π

Example

1. $\Pi = S/\sim_{AP}$
2. while some block $B \in \Pi$ is a splitter of Π
   2.1 pick a block B that is a splitter of Π
   2.2 $\Pi = \text{Refine}(\Pi, B)$
3. return Π

$(q, q') \in \sim_{AP} \Leftrightarrow L(q) = L(q')$

loop invariant: Π is coarser than $S/\sim_{TS}$



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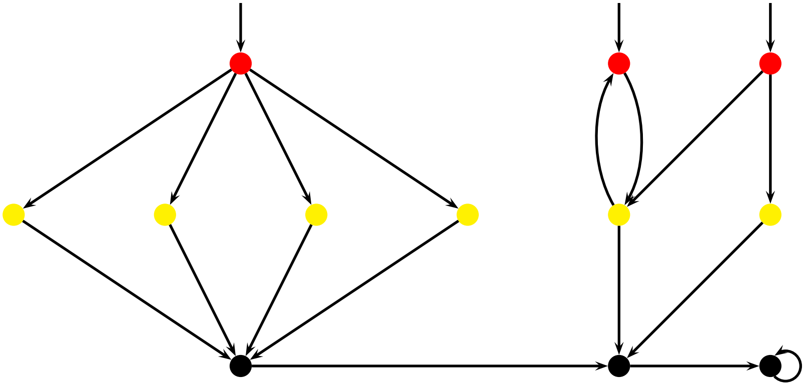

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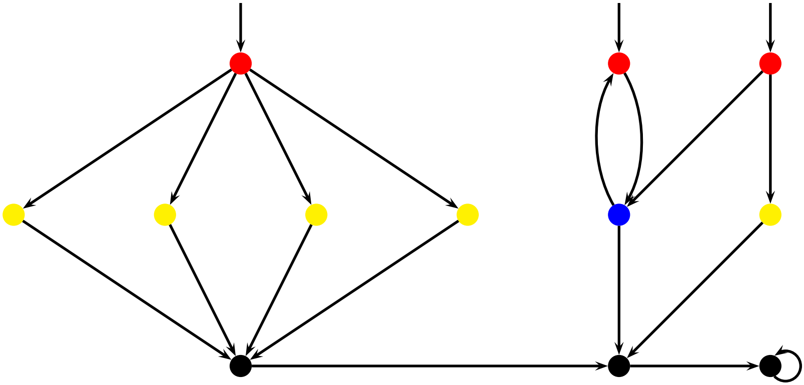

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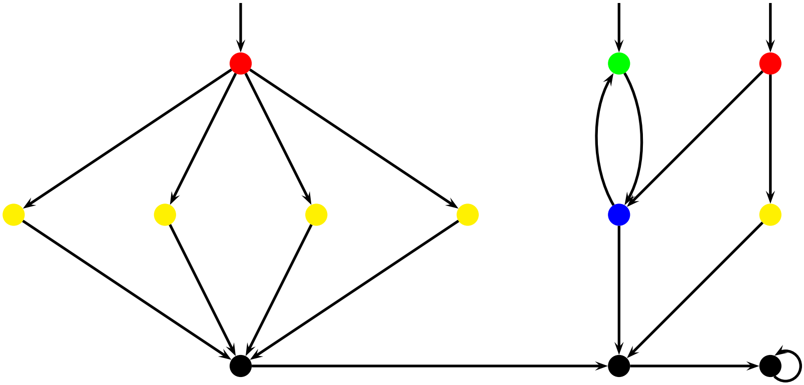

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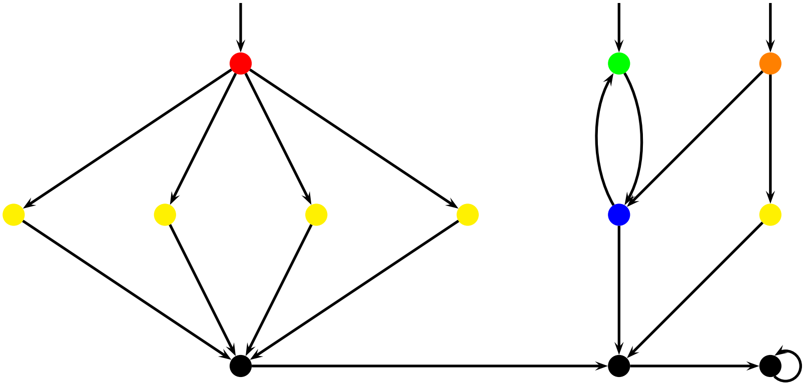
loop invariant: Π is coarser than $S/{\sim_{TS}}$



Correctness and termination

1. $\Pi = S/{\sim_{AP}}$ $\qquad (q, q') \in {\sim_{AP}} \Leftrightarrow L(q) = L(q')$
2. while some block $B \in \Pi$ is a splitter of Π $\qquad$ loop invariant: Π is coarser than $S/{\sim_{TS}}$
   2.1 pick a block B that is a splitter of Π
   2.2 $\Pi = \text{Refine}(\Pi, B)$
3. return Π

Lemma 3. The algorithm terminates.
Lemma 4. The loop invariant holds initially.
Lemma 5. The loop invariant is preserved.
Theorem. The algorithm returns the quotient $S/{\sim_{TS}}$ of the coarsest bisimulation ${\sim_{TS}}$.

Simulation order

Let $TS_i = (S_i, Act_i, \rightarrow_i, I_i, AP, L_i)$, i=1, 2, be two transition systems over AP.

A simulation for (TS_1, TS_2) is a binary relation $\mathcal{R} \subseteq S_1 \times S_2$ such that:

1. $\forall q_1 \in I_1 \; \exists q_2 \in I_2 . \; (q_1, q_2) \in \mathcal{R}$
2. for all $(q_1, q_2) \in \mathcal{R}$ it holds:
   2.1 $L_1(q_1) = L_2(q_2)$

   2.2 if $q_1' \in Post(q_1)$
   then there exists $q_2' \in Post(q_2)$ with $(q_1', q_2') \in \mathcal{R}$

$TS_1 \preceq TS_2$ iff there exists a simulation $\mathcal{R}$ for (TS_1, TS_2)

Simulation order

$$\begin{array}{ccc} q_1 & \to & q_1' \\ \mathcal{R} & & \\ q_2 & & \end{array} \quad \text{can be completed to} \quad \begin{array}{ccc} q_1 & \to & q_1' \\ \mathcal{R} & & \mathcal{R} \\ q_2 & \to & q_2' \end{array}$$

but not necessarily:

$$\begin{array}{ccc} q_1 & & \\ \mathcal{R} & & \\ q_2 & \to & q_2' \end{array} \quad \text{can be completed to} \quad \begin{array}{ccc} q_1 & \to & q_1' \\ \mathcal{R} & & \mathcal{R} \\ q_2 & \to & q_2' \end{array}$$

The use of simulations

- As a notion of correctness for refinement
  - $TS \preceq TS'$ whenever TS is obtained by deleting transitions from TS'
  - e.g., nondeterminism is resolved by choosing one alternative
- As a notion of correctness for abstraction
  - abstract from concrete values of certain program or control variables
  - use instead abstract values or ignore their value completely
  - used in e.g., software model checking of `C` and `Java`
  - formalized by an abstraction function f that maps s onto its abstraction $f(s)$

Abstraction function

- $f : S \to \widehat{S}$ is an abstraction function if
  $f(q) = f(q') \;\Rightarrow\; L(q) = L(q')$
  - S is a set of concrete states and $\widehat{S}$ a set of abstract states, i.e. $|\widehat{S}| \ll |S|$
- Abstraction functions are useful for:
  - data abstraction: abstract from values of program or control variables

    $$f : \text{concrete data domain} \to \text{abstract data domain}$$

  - predicate abstraction: use predicates over the program variables

    $$f : \text{state} \to \text{valuations of the predicates}$$

  - localization reduction: partition program variables into visible and invisible

    $$f : \text{all variables} \to \text{visible variables}$$

Abstract transition system

For $TS = (S, Act, \rightarrow, I, AP, L)$ and abstraction function $f : S \rightarrow \widehat{S}$ let:

$$TS_f \;=\; (\widehat{S}, Act, \rightarrow_f, I_f, AP, L_f), \qquad \text{the abstraction of } TS \text{ under } f$$

where

- $\rightarrow_f$ is defined by: $$\frac{s \xrightarrow{\alpha} s'}{f(s) \xrightarrow{\alpha}_f f(s')}$$
- $I_f = \{ f(s) \mid s \in I \}$
- $L_f(f(s)) = L(s)$; for $s \in \widehat{S} \setminus f(S)$, labeling is undefined

$\mathcal{R} = \{ (s, f(s)) \mid s \in S \}$ is a simulation for (TS, TS_f)

Simulation order on paths

Whenever we have:

$$\begin{array}{ccccccccc} s_0 & \rightarrow & s_1 & \rightarrow & s_2 & \rightarrow & s_3 & \rightarrow & s_4 \ldots\ldots \\ \mathcal{R} & & & & & & & & \\ t_0 & & & & & & & & \end{array}$$

this can be completed to

$$\begin{array}{ccccccccc} s_0 & \rightarrow & s_1 & \rightarrow & s_2 & \rightarrow & s_3 & \rightarrow & s_4 \ldots\ldots \\ \mathcal{R} & & \mathcal{R} & & \mathcal{R} & & \mathcal{R} & & \mathcal{R} \\ t_0 & \rightarrow & t_1 & \rightarrow & t_2 & \rightarrow & t_3 & \rightarrow & t_4 \ldots\ldots \end{array}$$

the proof of this fact is by induction on the length of the path

Simulation is a pre-order

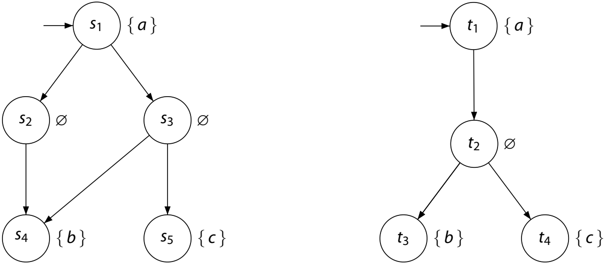
$\preceq$ is a preorder, i.e., reflexive and transitive

Simulation equivalence

TS_1 and TS_2 are simulation equivalent, denoted $TS_1 \simeq TS_2$,

if $TS_1 \preceq TS_2$ and $TS_2 \preceq TS_1$

Similar but not bisimilar



$TS_{left} \simeq TS_{right}$ but $TS_{left} \not\sim TS_{right}$

Simulation order on states

A simulation for $TS = (S, Act, \rightarrow, I, AP, L)$ is a binary relation $\mathcal{R} \subseteq S \times S$ such that for all $(q_1, q_2) \in \mathcal{R}$:

1. $L(q_1) = L(q_2)$
2. if $q_1' \in Post(q_1)$
   then there exists an $q_2' \in Post(q_2)$ with $(q_1', q_2') \in \mathcal{R}$

q_1 is simulated by q_2, denoted by $q_1 \preceq_{TS} q_2$,
if there exists a simulation $\mathcal{R}$ for TS with $(q_1, q_2) \in \mathcal{R}$

$$q_1 \preceq_{TS} q_2 \quad \text{if and only if} \quad TS_{q_1} \preceq TS_{q_2}$$

$$q_1 \simeq_{TS} q_2 \quad \text{if and only if} \quad q_1 \preceq_{TS} q_2 \text{ and } q_2 \preceq_{TS} q_1$$

Simulation quotient

For $TS = (S, Act, \rightarrow, I, AP, L)$ and simulation equivalence $\simeq \subseteq S \times S$ let

$$TS/\simeq \;=\; (S', \{\,\tau\,\}, \rightarrow', I', AP, L'),$$ the quotient of TS under $\simeq$

where

- $S' = S/\simeq = \{\, [s]_\simeq \mid s \in S \,\}$ and $I' = \{\, [s]_\simeq \mid s \in I \,\}$
- $\rightarrow'$ is defined by:
- $L'([s]_\simeq) = L(s)$

$$\frac{s \xrightarrow{\alpha} s'}{[s]_\simeq \xrightarrow{\tau}{}' [s']_\simeq}$$

lemma: $TS \simeq TS/\simeq$; proof not straightforward!

Universal fragment of CTL*

$\forall$CTL* state-formulas are formed according to:

$$\Phi ::= \text{true} \;\Big|\; \text{false} \;\Big|\; a \;\Big|\; \neg a \;\Big|\; \Phi_1 \wedge \Phi_2 \;\Big|\; \Phi_1 \vee \Phi_2 \;\Big|\; \mathsf{A}\,\varphi$$

where $a \in AP$ and φ is a path-formula

$\forall$CTL* path-formulas are formed according to:

$$\varphi ::= \Phi \;\Big|\; \mathsf{X}\,\varphi \;\Big|\; \varphi_1 \wedge \varphi_2 \;\Big|\; \varphi_1 \vee \varphi_2 \;\Big|\; \varphi_1 \,\mathsf{U}\, \varphi_2 \;\Big|\; \varphi_1 \,\mathsf{R}\, \varphi_2$$

where Φ is a state-formula, and φ, φ_1 and φ_2 are path-formulas

Universal CTL* contains LTL

For every LTL formula there exists an equivalent $\forall$CTL* formula

Proof: Bring LTL formula into positive normal form (PNF).

Simulation order and $\forall CTL^*$

Let TS be a finite transition system (without terminal states) and q, q' states in TS.

The following statements are equivalent:

(1) $q \preceq_{TS} q'$

(2) for all $\forall CTL^*$-formulas Φ: $q' \models \Phi$ implies $q \models \Phi$

(3) for all $\forall CTL$-formulas Φ: $q' \models \Phi$ implies $q \models \Phi$

proof is carried out in three steps: (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1)

Existential fragment of CTL*

$\exists$CTL* state-formulas are formed according to:

$$\Phi ::= \text{true} \;\Big|\; \text{false} \;\Big|\; a \;\Big|\; \neg a \;\Big|\; \Phi_1 \wedge \Phi_2 \;\Big|\; \Phi_1 \vee \Phi_2 \;\Big|\; \exists \varphi$$

where $a \in AP$ and φ is a path-formula

$\exists$CTL* path-formulas are formed according to:

$$\varphi ::= \Phi \;\Big|\; \mathsf{X}\,\varphi \;\Big|\; \varphi_1 \wedge \varphi_2 \;\Big|\; \varphi_1 \vee \varphi_2 \;\Big|\; \varphi_1 \,\mathsf{U}\, \varphi_2 \;\Big|\; \varphi_1 \,\mathsf{R}\, \varphi_2$$

where Φ is a state-formula, and φ, φ_1 and φ_2 are path-formulas

Simulation order and $\exists CTL^*$

Let *TS* be a finite transition system (without terminal states) and q, q' states in *TS*.

The following statements are equivalent:

(1) $q \preceq_{TS} q'$

(2) for all $\exists CTL^*$-formulas Φ: $q \models \Phi$ implies $q' \models \Phi$

(3) for all $\exists CTL$-formulas Φ: $q \models \Phi$ implies $q' \models \Phi$

$\simeq$, $\forall CTL^*$, and $\exists CTL^*$ equivalence

For finite transition system *TS* without terminal states:

$$\simeq_{TS} \;=\; \equiv_{\forall CTL^*} \;=\; \equiv_{\forall CTL} \;=\; \equiv_{\exists CTL^*} \;=\; \equiv_{\exists CTL}$$

Skeleton for simulation preorder checking

Require: finite transition system $TS = (S, Act, \to, I, AP, L)$ over AP
Ensure: simulation order $\preceq_{TS}$

$\mathcal{R} := \{ (q_1, q_2) \mid L(q_1) = L(q_2) \};$

while $\mathcal{R}$ is not a simulation **do**
  choose $(q_1, q_2) \in \mathcal{R}$
    such that $(q_1, q_1') \in E$, but for all q_2' with $(q_2, q_2') \in E$, $(q_1', q_2') \notin \mathcal{R}$;
  $\mathcal{R} := \mathcal{R} \setminus \{ (q_1, q_2) \}$
end while
return $\mathcal{R}$

The number of iterations is bounded above by $|S|^2$, since:

$$Q \times Q \supseteq \mathcal{R}_0 \supsetneqq \mathcal{R}_1 \supsetneqq \mathcal{R}_2 \supsetneqq \ldots \supsetneqq \mathcal{R}_n = \preceq$$