Dr. Martin Zimmermann
Leander Tentrup, M.Sc.

Summer term 2015
Problem Set 4
Due: Tuesday, May 26[th], 2015

Recursion Theory

Problem 1: Encodings 2 + 1 + 1 + 2 + 2 Points

Recall: $[x, y] = 2^x(2y + 1) - 1$ and the inverses $[\cdot]_1, [\cdot]_2 \colon \mathbb{N} \to \mathbb{N}$ satisfy $[[x]_1, [x]_2] = x$ for all x.

Also, $\langle \varepsilon \rangle = 0$ and $\langle x_0, \ldots, x_k \rangle = p_0^{x_0} \cdots p_{k-1}^{x_{k-1}} p_k^{x_k+1} - 1$, where p_i is the i-th prime number.

Show that the following functions are primitive recursive:

1. $(x)_y$ = the exponent of the y-th prime number p_y in the prime factorization of x. Convention: $(0)_y = 0$ and $(1)_y = 0$ for every y.
2. The pairing function $[\cdot, \cdot]$.
3. The inverses $[\cdot]_1$ and $[\cdot]_2$ of the pairing function.
4. $\text{len}(x) = \begin{cases} k+1 & \text{if } x = \langle x_0, \ldots, x_k \rangle, \\ 0 & \text{if } x = 0. \end{cases}$
5. $\langle x \rangle_y = \begin{cases} x_y & \text{if } x = \langle x_0, \ldots, x_k \rangle \text{ and } y \leq k, \\ 0 & \text{otherwise.} \end{cases}$

Problem 2: Course-of-Values Recursion 3 + 1 Points

1. Recall the scheme of course-of-values recursion: given $g \colon \mathbb{N}^n \to \mathbb{N}$ and $h \colon \mathbb{N}^{n+2} \to \mathbb{N}$ define $f \colon \mathbb{N}^{n+1} \to \mathbb{N}$ by

   $$f(\overline{x}, 0) = g(\overline{x}) \qquad \text{and} \qquad f(\overline{x}, S(y)) = h(\overline{x}, y, \langle f(\overline{x}, 0), \ldots, f(\overline{x}, y) \rangle).$$

   Prove: if g and h are primitive recursive, then so is f.
2. The Fibonacci function $\text{fib} \colon \mathbb{N} \to \mathbb{N}$ is given by the recursion $\text{fib}(0) = \text{fib}(1) = 1$ and $\text{fib}(n) = \text{fib}(n-1) + \text{fib}(n-2)$ for $n > 1$.

   Show that fib is primitive recursive.

Hint: For 1., define a suitable auxiliary function.