Chapter 1

Many-Sorted Logic

Decision Procedures
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1.1 Syntax

1.1.1 Definition
We fix an enumerable set **Sort** of sorts.

1.1.2 Definition
We fix an enumerable set **Var** of variables. Each variable has associated to it a sort. We denote with $\mathbf{Var}_\sigma$ the set of variables of sort σ. We assume that $\mathbf{Var}_\sigma$ is enumerable, for all sorts σ.

1.1.3 Definition
We fix an enumerable set **Con** of constant symbols. Each constant symbol has associated to it a sort. We denote with $\mathbf{Con}_\sigma$ the set of constant symbols of sort σ. We assume that $\mathbf{Con}_\sigma$ is enumerable, for all sorts σ.

1.1.4 Definition
We fix an enumerable set **Fun** of function symbols. Each function symbol has associated to it an arity of the form $\sigma_1 \times \cdots \times \sigma_n \to \sigma$, where $n \geq 1$ and $\sigma_1, \ldots, \sigma_n, \sigma$ are sorts. We denote with $\mathbf{Fun}_{\sigma_1 \times \cdots \times \sigma_n \to \sigma}$ the set of function symbols of arity $\sigma_1 \times \cdots \times \sigma_n \to \sigma$. We assume that $\mathbf{Fun}_{\sigma_1 \times \cdots \times \sigma_n \to \sigma}$ is enumerable, for all sorts $\sigma_1, \ldots, \sigma_n, \sigma$.

1.1.5 Definition
We fix an enumerable set **Pred** of predicate symbols. Each predicate symbol has associated to it an arity of the form $\sigma_1 \times \cdots \times \sigma_n$, where $n \geq 1$ and $\sigma_1, \ldots, \sigma_n$ are sorts. We denote with $\mathbf{Pred}_{\sigma_1 \times \cdots \times \sigma_n}$ the set of predicate symbols of arity $\sigma_1 \times \cdots \times \sigma_n$. We assume that $\mathbf{Pred}_{\sigma_1 \times \cdots \times \sigma_n}$ is enumerable, for all sorts $\sigma_1, \ldots, \sigma_n$.

1.1.6 Definition

The EQUALITY SYMBOL is $\approx$.

1.1.7 Definition

The PROPOSITIONAL CONNECTIVES are

1. $\neg$ (not);
2. $\wedge$ (and);
3. $\vee$ (or);
4. $\rightarrow$ (implies);
5. $\leftrightarrow$ (iff).

1.1.8 Definition

The UNIVERSAL QUANTIFIER is $\forall$.

1.1.9 Definition

The EXISTENTIAL QUANTIFIER is $\exists$.

1.1.10 Definition

A SIGNATURE is a tuple $\Sigma = (S, C, F, P)$ where:

1. S is a nonempty set of sorts.
2. C is a countable set of constant symbols whose sorts belong to S.
3. F is a countable set of function symbols whose arities are constructed using sorts that belong to S.
4. P is a countable set of predicate symbols whose arities are constructed using sorts that belong to S.

Given a signature $\Sigma = (S, C, F, P)$, we write Σ^{S} for S, Σ^{C} for C, Σ^{F} for F, and Σ^{P} for P.

1.1.11 Definition

Let Σ be a signature. The set of Σ-TERMS of sort σ is the smallest set of expressions satisfying the following properties:

- Each variable x of sort σ is a term of sort σ, provided that $\sigma \in \Sigma^{\mathrm{S}}$.
- Each constant symbol $c \in \Sigma^{\mathrm{C}}$ of sort σ is a Σ-term of sort σ.
- If $f \in \Sigma^{\mathrm{F}}$ is a function symbol of arity $\sigma_1 \times \cdots \times \sigma_n \rightarrow \sigma$ and t_i is a Σ-term of sort σ_i, for $i = 1, \dots, n$, then $f(t_1, \dots, t_n)$ is a term of sort σ.

1.1.12 Definition

Let Σ be a signature. A Σ-ATOM is an expressions of the form

$$s \approx t\,, \qquad p(t_1, \dots, t_n) \qquad ,$$

where:

1. s and t are Σ-terms of the same sort;
2. $p \in \Sigma^{\mathrm{P}}$ is a predicate symbol of arity $\sigma_1 \times \cdots \times \sigma_n$ and t_i is a Σ-term of sort σ_i, for $i = 1, \ldots, n$.

1.1.13 Definition
The set of Σ-FORMULAE is the smallest set of expressions satisfying the following properties:

1. Each Σ-atom is a Σ-formula.
2. If φ is a Σ-formula then $\neg\varphi$ is a Σ-formula.
3. If φ and ψ are Σ-formulae then $\varphi \wedge \psi$, $\varphi \vee \psi$, $\varphi \rightarrow \psi$, and $\varphi \leftrightarrow \psi$ are formulae.
4. If φ is a Σ-formula, $\sigma \in \Sigma^{\mathrm{S}}$, and x is a variable of sort σ, then $(\forall_\sigma x)\varphi$ and $(\exists_\sigma x)\varphi$ are Σ-formulae.

1.1.14 Definition
A Σ-LITERAL is a formula of the form

$$\varphi\,, \qquad\qquad \neg\varphi\,,$$

where φ is a Σ-atom.

1.1.15 Definition
A QUANTIFIER-FREE Σ-FORMULA is a Σ-formula in which no quantifier occurs.

1.1.16 Definition
Let t be a term, and let σ be a sort. We denote with $vars_\sigma(t)$ the set of variables of sort σ occurring in t. This set can be recursively defined as follows:

1. $vars_\sigma(x) = \{x\}$, for all variables x of sort σ.
2. $vars_\sigma(x) = \varnothing$, for all variables x whose sort is not σ.
3. $vars_\sigma(c) = \varnothing$, for all constant symbols c.
4. $vars_\sigma(f(t_1, \ldots, t_n)) = \bigcup_{i=1}^n vars_\sigma(t_i)$.

1.1.17 Definition
Let t be a term. We denote with $vars(t)$ the set of variables occurring in t, that is,

$$vars(t) = \bigcup_{\sigma \in \mathbf{Sort}} vars_\sigma(t)\,.$$

1.1.18 Definition
Let T be a set of terms. We let

$$vars_\sigma(T) = \bigcup_{t \in T} vars_\sigma(t)\,,$$

1.1.19 Definition
Let T be a set of terms. We let

$$vars(T) = \bigcup_{t \in T} vars(t)\,.$$

1.1.20 Definition
Let φ be a formula, and let σ be a sort. We denote with $vars_\sigma(\varphi)$ the set of variables occurring free in φ. This set can be recursively defined as follows:

1. $vars_\sigma(s \approx t) = vars_\sigma(s) \cup vars_\sigma(t)$.
2. $vars_\sigma(p(t_1, \ldots, t_n)) = \bigcup_{i=1}^{n} vars_\sigma(t_i)$.
3. $vars_\sigma(\neg\varphi_1) = vars_\sigma(\varphi_1)$.
4. $vars_\sigma(\varphi_1 \wedge \varphi_2) = vars_\sigma(\varphi_1) \cup vars_\sigma(\varphi_2)$.
5. $vars_\sigma(\varphi_1 \vee \varphi_2) = vars_\sigma(\varphi_1) \cup vars_\sigma(\varphi_2)$.
6. $vars_\sigma(\varphi_1 \rightarrow \varphi_2) = vars_\sigma(\varphi_1) \cup vars_\sigma(\varphi_2)$.
7. $vars_\sigma(\varphi_1 \leftrightarrow \varphi_2) = vars_\sigma(\varphi_1) \cup vars_\sigma(\varphi_2)$.
8. $vars_\sigma((\forall_\tau x)\varphi_1) = vars_\sigma(\varphi_1) \setminus \{x\}$.
9. $vars_\sigma((\exists_\tau x)\varphi_1) = vars_\sigma(\varphi_1) \setminus \{x\}$.

1.1.21 Definition
Let φ be a formula. We denote with $vars(\varphi)$ the set of variables occurring free in φ, that is,

$$vars(\varphi) = \bigcup_{\sigma \in \mathbf{Sort}} vars_\sigma(\varphi)\,.$$

1.1.22 Definition
Let Φ be a set of formulae. We let

$$vars_\sigma(\Phi) = \bigcup_{\varphi \in \Phi} vars_\sigma(\varphi)\,,$$

1.1.23 Definition
Let Φ be a set of formulae. We let

$$vars(\Phi) = \bigcup_{\varphi \in \Phi} vars(\varphi)\,.$$

1.1.24 Definition
Let Σ be a signature. A Σ-SENTENCE is a Σ-formula φ such that $vars(\varphi) = \varnothing$.

1.2 Semantics

1.2.1 Definition
Let Σ be a signature, and let X be a set of variables whose sorts are in Σ^{S}. A Σ-INTERPRETATION over X is a map satisfying the following properties:

1. Each sort $\sigma \in \Sigma^{\mathrm{S}}$ is mapped to a nonempty domain A_σ.
2. Each variable $x \in X$ of sort σ is mapped to an element $x^{\mathcal{A}} \in A_\sigma$.
3. Each constant symbol $c \in \Sigma^{\mathrm{C}}$ of sort σ is mapped to an element $c^{\mathcal{A}} \in A_\sigma$.
4. Each function symbol $f \in \Sigma^{\mathrm{F}}$ of arity $\sigma_1 \times \cdots \times \sigma_n \to \sigma$ is mapped to a function $f^{\mathcal{A}} : A_{\sigma_1} \times \cdots \times A_{\sigma_n} \to A_\sigma$.
5. Each predicate symbol $p \in \Sigma^{\mathrm{P}}$ of arity $\sigma_1 \times \cdots \times \sigma_n$ is mapped to a subset $p^{\mathcal{A}} \subseteq A_{\sigma_1} \times \cdots \times A_{\sigma_n}$.

1.2.2 Definition
Let Σ be a signature. A Σ-STRUCTURE is a Σ-interpretation over an empty set of variables.

1.2.3 Definition
Let Σ be a signature, let t be a Σ-term of sort σ, and let $\mathcal{A}$ be a Σ-interpretation over X such that $\mathit{vars}(t) \subseteq X$. The EVALUATION of t under $\mathcal{A}$ is the object $t^{\mathcal{A}} \in A_\sigma$ recursively defined as follows:

1. The evaluation of a variable x is $x^{\mathcal{A}}$.
2. The evaluation of a constant symbol c is $c^{\mathcal{A}}$.
3. The evaluation of a term $f(t_1, \ldots, t_n)$ is

$$[f(t_1, \ldots, t_n)]^{\mathcal{A}} = f^{\mathcal{A}}(t_1^{\mathcal{A}}, \ldots, t_n^{\mathcal{A}})\,.$$

1.2.4 Definition
Let $\mathcal{A}$ and $\mathcal{B}$ be Σ-interpretations over X, and let $x \in X$ be a variable. We say that $\mathcal{B}$ is an x-VARIANT of $\mathcal{A}$ if:

1. $A_\sigma = B_\sigma$, for all sorts $\sigma \in \Sigma^{\mathrm{S}}$.
2. $r^{\mathcal{A}} = r^{\mathcal{B}}$, for all objects $r \in \Sigma^{\mathrm{C}} \cup \Sigma^{\mathrm{F}} \cup \Sigma^{\mathrm{P}} \cup (X \setminus \{x\})$.

1.2.5 Definition
Let Σ be a signature, let φ be a Σ-formula, and let $\mathcal{A}$ be a Σ-interpretation over X such that $\mathit{vars}(\varphi) \subseteq X$. The EVALUATION of φ under $\mathcal{A}$ is the truth value $\varphi^{\mathcal{A}} \in A_\sigma$ recursively defined as follows:

1. $[s \approx t]^{\mathcal{A}} = \mathit{true} \iff s^{\mathcal{A}} = t^{\mathcal{A}}$.
2. $[p(t_1, \ldots, t_n)]^{\mathcal{A}} = \mathit{true} \iff (t_1^{\mathcal{A}}, \ldots, t_n^{\mathcal{A}}) \in p^{\mathcal{A}}$.

3. $[\neg\varphi]^{\mathcal{A}} = \mathit{true} \iff \varphi^{\mathcal{A}} = \mathit{false}$.
4. $[\varphi \wedge \psi]^{\mathcal{A}} = \mathit{true} \iff \varphi^{\mathcal{A}} = \mathit{true}$ and $\psi^{\mathcal{A}} = \mathit{true}$.
5. $[\varphi \vee \psi]^{\mathcal{A}} = \mathit{true} \iff \varphi^{\mathcal{A}} = \mathit{true}$ or $\psi^{\mathcal{A}} = \mathit{true}$.
6. $[\varphi \to \psi]^{\mathcal{A}} = \mathit{true} \iff \varphi^{\mathcal{A}} = \mathit{false}$ or $\psi^{\mathcal{A}} = \mathit{true}$.
7. $[(\forall_\sigma x)\varphi]^{\mathcal{A}} = \mathit{true} \iff$
$$\varphi^{\mathcal{B}} = \mathit{true}\,, \qquad \text{for all } x\text{-variants } \mathcal{B} \text{ of } \mathcal{A}\,.$$
8. $[(\exists_\sigma x)\varphi]^{\mathcal{A}} = \mathit{true} \iff$
$$\varphi^{\mathcal{B}} = \mathit{true}\,, \qquad \text{for some } x\text{-variant } \mathcal{B} \text{ of } \mathcal{A}\,.$$

1.2.6 Definition
Let $\mathcal{A}$ be a Σ-interpretation over X, and let φ be a Σ-formula such that $\mathit{vars}(\varphi) \subseteq X$. We write
$$\mathcal{A} \models \varphi$$
when $\varphi^{\mathcal{A}} = \mathit{true}$.

1.2.7 Definition
Let φ be a Σ-formula, and let $X = \mathit{vars}(\varphi)$. We say that φ is:

- VALID, if $\mathcal{A} \models \varphi$, for all Σ-interpretations $\mathcal{A}$ over X;
- SATISFIABLE, if $\mathcal{A} \models \varphi$, for some Σ-interpretation $\mathcal{A}$ over X;
- UNSATISFIABLE, if φ is not satisfiable.

1.2.8 Definition
Let $\mathcal{A}$ be a Σ-interpretation over X, and let Φ be a set of Σ-formulae such that $\mathit{vars}(\Phi) \subseteq X$. We write
$$\mathcal{A} \models \Phi$$
when
$$\mathcal{A} \models \varphi\,, \qquad \text{for all formulae } \varphi \in \Phi\,.$$

1.2.9 Definition
Let Φ be a set of Σ-formulae, and let $X = \mathit{vars}(\Phi)$. We say that Φ is:

- VALID, if $\mathcal{A} \models \Phi$, for all Σ-interpretations $\mathcal{A}$ over X;
- SATISFIABLE, if $\mathcal{A} \models \Phi$, for some Σ-interpretation $\mathcal{A}$ over X;
- UNSATISFIABLE, if Φ is not satisfiable.

1.2.10 Definition
Let $\mathcal{A}$ be a Σ-interpretation over X. For $\Sigma_0 \subseteq \Sigma$ and $X_0 \subseteq X$, we denote with $\mathcal{A}^{\Sigma_0,X_0}$ the interpretation obtained from $\mathcal{A}$ by restricting it to interpret only the symbols in Σ_0 and the variables in X_0. Furthermore, we let $\mathcal{A}^{\Sigma_0} = \mathcal{A}^{\Sigma_0,\varnothing}$.

1.2.11 Definition
Let $\mathcal{A}$ and $\mathcal{B}$ be two Σ-interpretations over X. An ISOMORPHISM h of $\mathcal{A}$ into $\mathcal{B}$ is a family of bijective functions

$$h = \left\{ h_\sigma : A_\sigma \to B_\sigma \mid \sigma \in \Sigma^{\mathrm{S}} \right\}$$

such that:

1. $h_\sigma(x^{\mathcal{A}}) = x^{\mathcal{B}}$, for all variables $x \in X_\sigma$.
2. $h_\sigma(c^{\mathcal{A}}) = c^{\mathcal{B}}$, for all constant symbols $c \in \Sigma^{\mathrm{C}}$.
3. $h_\sigma(f^{\mathcal{A}}(a_1, \dots, a_n)) = f^{\mathcal{B}}(h_{\sigma_1}(a_1), \dots, h_{\sigma_n}(a_n))$, for all function symbol $f \in \Sigma^{\mathrm{F}}$ of arity $\sigma_1 \times \cdots \times \sigma_n \to \sigma$.
4. $(a_1, \dots, a_n) \in p^{\mathcal{A}}$ if and only if $(h_{\sigma_1}(a_1), \dots, h_{\sigma_n}(a_n)) \in p^{\mathcal{B}}$, for all predicate symbol $p \in \Sigma^{\mathrm{P}}$ of arity $\sigma_1 \times \cdots \times \sigma_n$.

We write $\mathcal{A} \cong \mathcal{B}$ when there is an isomorphism of $\mathcal{A}$ into $\mathcal{B}$.

1.3 Modelclasses

1.3.1 Definition
A Σ-MODELCLASS is a pair $M = (\Sigma, \mathbf{A})$ such that:

1. Σ is a signature;
2. $\mathbf{A}$ is a class of Σ-structures;
3. $\mathbf{A}$ is closed under isomorphism.

1.3.2 Definition
Let $M = (\Sigma, \mathbf{A})$ be a modelclass. An M-STRUCTURE is a Σ-structure $\mathcal{A}$ such that $\mathcal{A} \in \mathbf{A}$.

1.3.3 Definition
Let $M = (\Sigma, \mathbf{A})$ be a modelclass. An M-INTERPRETATION is a Σ-interpretation $\mathcal{A}$ such that $\mathcal{A}^{\Sigma}$ is a Σ-structure.

1.3.4 Definition
Let M be a Σ-modelclass, let $\mathcal{A}$ be a Σ-interpretation over X, and let φ be a Σ-formula such that $\mathit{vars}(\varphi) \subseteq X$. We write

$$\mathcal{A} \models_M \varphi\,,$$

whenever $\varphi^{\mathcal{A}} = \mathit{true}$ and $\mathcal{A}^{\Sigma}$ is an M-structure.

1.3.5 Definition
Let M be a Σ-modelclass, let φ be a Σ-formula, and let $X = \mathit{vars}(\varphi)$. We say that φ is:

- M-VALID, if $\mathcal{A} \models_M \varphi$, for all M-interpretations $\mathcal{A}$ over X;
- M-SATISFIABLE, if $\mathcal{A} \models_M \varphi$, for some M-interpretation $\mathcal{A}$ over X;
- M-UNSATISFIABLE, if φ is not M-satisfiable.

1.3.6 Definition
Let M be a Σ-modelclass, let $\mathcal{A}$ be a Σ-interpretation over X, and let Φ be a set of Σ-formulae such that $\mathit{vars}(\Phi) \subseteq X$. We write

$$\mathcal{A} \models_M \Phi$$

when

$$\mathcal{A} \models_M \varphi\,, \qquad \text{for all formulae } \varphi \in \Phi\,.$$

1.3.7 Definition
Let M be a Σ-modelclass, let Φ be a set of Σ-formulae, and let $X = \mathit{vars}(\Phi)$. We say that Φ is:

- M-VALID, if $\mathcal{A} \models_M \Phi$, for all Σ-interpretations $\mathcal{A}$ over X;
- M-SATISFIABLE, if $\mathcal{A} \models_M \Phi$, for some Σ-interpretation $\mathcal{A}$ over X;
- M-UNSATISFIABLE, if Φ is not M-satisfiable.

1.3.8 Definition
Let M be a Σ-modelclass, and let L be a set of Σ-formulae. We define the following decision problems:

- The VALIDITY PROBLEM of M with respect to L is the problem of deciding, for each Σ-formula $\varphi \in L$, whether or not φ is M-valid.
- The SATISFIABILITY PROBLEM of M with respect to L is the problem of deciding, for each Σ-formula $\varphi \in L$, whether or not φ is M-satisfiable.
- The UNSATISFIABILITY PROBLEM of M with respect to L is the problem of deciding, for each Σ-formula $\varphi \in L$, whether or not φ is M-unsatisfiable.

When we mention a decision problem without specifying the set of formulae L, we implicitly assume that L is the set of all Σ-formulae. For instance, if M is a Σ-modelclass, the validity problem of a Σ-model class M is the problem of deciding, for each Σ-formula φ whether or not φ is M-valid.

When we prefix the name of a decision problem with "quantifier-free", we implicitly assume that L is the set of all quantifier-free Σ-formulae. For instance, the quantifier-free satisfiability problem of a Σ-model class M is the problem of deciding, for each quantifier-free Σ-formula φ whether or not φ is M-satisfiable.