Bernd Finkbeiner Date: January 29, 2013

Automata, Games, and Verification: Lecture 14

Corollary 1 *S2S is decidable.*

SnS is the monadic second order theory of n successors.

Corollary 2 *SnS is decidable.*

Proof:

Repeat exercise for automata on n-ary trees. ■

SωS is the monadic second order theory of ω successors.

Theorem 1 *SωS is decidable.*

Proof:

We give an effective translation from SωS to S2S.

- Bijection β from ω^* to $0\mathbb{B}^*$:
  - $\beta(\varepsilon) := \varepsilon$
  - $\beta(vn) := \beta(v)01^n$
- One-to-many relation R between SωS and S2S structures: label a position $\beta(x)$ in the binary tree with σ iff x is labeled with σ in the ω-ary tree.
- Bring given SωS formula in normal form and translate as follows:
  - $x = \varepsilon \mapsto x = \varepsilon$
  - $x = yn \mapsto x = y01^n$ for $n \in \omega$
  - $x \in Y \mapsto x \in Y$
  - $x = y \mapsto x \in Y$
  - $\exists X \ldots \mapsto \exists X .(\forall y \in X . \neg 1 \leq y) \land \ldots$

■

19 Alternating Tree Automata

Definition 1 *An alternating tree automaton over binary Σ-trees is a tuple $\mathcal{A} = (S, s_0, \delta, \varphi)$:*

- *S: finite set of states*
- $s_0 \in S$
- $\delta : S \times \Sigma \to \mathbb{B}^+(\{0,1\} \times S)$ *is the transition function.*
- φ*: acceptance condition (Büchi, parity, ...)*

More general: set of directions $\mathcal{D} = \{0, \dots, k-1\}$, $T \subseteq \mathcal{D}^*$, degree $d : \mathcal{D}^* \to \{1, \dots, k\}$

Definition 2 *An alternating tree automaton over Σ-trees is a tuple $\mathcal{A} = (S, s_0, \delta, \varphi)$:*

- *S: finite set of states*
- $s_0 \in S$
- $\delta : S \times \Sigma \times \{1, \dots k\} \to \mathbb{B}^+(\{0, 1, \dots k-1\} \times S)$ *is the transition function.*
- φ*: acceptance condition (Büchi, parity, ...)*

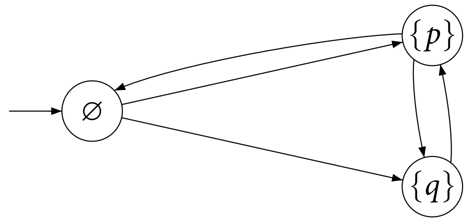
Definition 3 *A run of a tree automaton $\mathcal{A}$ on a Σ-tree v is a $\mathcal{D}^* \times S$-tree (T, r), s.t.*

1. $r(\varepsilon) = (\varepsilon, s_0)$
2. *Let $y \in T$ with $r(y) = (x, q)$ and $\delta(q, v(x), d(x)) = \theta$. Then there is a (possibly empty) set $Q = \{(c_0, q'_0), (c_1, q'_1), \dots, (c_n, q'_n)\} \subseteq \{0, \dots, d(x) - 1\} \times S$, such that the following hold:*
   - $Q \models \theta$
   - *for all $0 \le i \le n$, we have $y \cdot i \in T$ and $r(y \cdot i) = (x \cdot c_i, q'_i)$.*

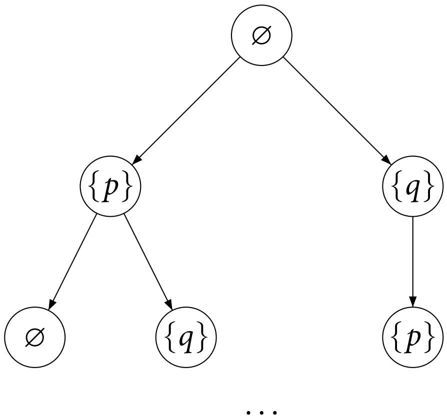
Definition 4 *A run is accepting if every branch is accepting (by φ). A Σ-tree is accepted if there exists an accepting run.*

Tree automata on Transition Systems

Example: For a transition system:



we build a computation tree t



Let k be the max number of successors in the transition system $(AP, S, s_0, \to, L)$. Define a mapping: $f : \{0, \dots, k-1\}^* \to S$:

- $f(\varepsilon) = s_o$
- Assume there is, for each $s \in S$, a fixed order on the successors $s'_1, s'_2, \dots$ of s
  $f(w \cdot i) = s'_i$ where s'_i is the ith successor of $s = f(w)$.

Definition 5 *The* computation tree *of a transition system* $(AP, S, s_0, \to, L)$ *is a* 2^{AP}*-tree* (T, t) *with* $t(v) = L(f(v))$ *and* $d(v) = d(f(v))$ *for all* $v \in T$.

Theorem 2 *The computation tree of a transition system is accepted by an alternating tree automaton* $\mathcal{A} = (S_\mathcal{A}, s_0, \delta, \varphi)$ *iff Player 0 has a winning strategy from* (s_0, q_0) *in the following game:*

- $V_0 = S_\mathcal{A} \times S_\mathcal{M}$
- $V_1 = S_\mathcal{A} \times 2^{\{0,\dots,k-1\} \times S_\mathcal{A}} \times S_\mathcal{M}$
- $E = \{((s, q), (s, \eta, q)) \mid \eta \models \delta(s, L(q), d(q))\}$
  $\cup \{((s, \eta, q), (s', q')) \mid (i, q') \in \eta, s'$ *is the* i*th successor of* $s\}$
- *winning condition:* φ *applied to the first component*

CTL

Translation from CTL formula φ to alternating Büchi tree automaton $\mathcal{A}_\varphi$:

- S =closure(φ):= set of all subformulas and their negations
- for $p \in AP$:
  - $\delta(p, \sigma, k) = true$ if $p \in \sigma$

- $\delta(p, \sigma, k) = \mathit{false}$ if $p \notin \sigma$
- $\delta(\neg p, \sigma, k) = \mathit{false}$ if $p \in \sigma$
- $\delta(\neg p, \sigma, k) = \mathit{true}$ if $p \notin \sigma$

- $\delta(\varphi \wedge \psi, \sigma, k) = \delta(\varphi, \sigma, k) \wedge \delta(\psi, \sigma, k)$
- $\delta(\varphi \vee \psi, \sigma, k) = \delta(\varphi, \sigma, k) \vee \delta(\psi, \sigma, k)$
- $\delta(\mathrm{AX}\varphi, \sigma, k) = \bigwedge_{c=0}^{k-1}(c, \varphi)$
- $\delta(\mathrm{EX}\varphi, \sigma, k) = \bigvee_{c=0}^{k-1}(c, \varphi)$
- $\delta(\mathrm{A}\varphi\,\mathcal{U}\,\psi, \sigma, k) = \delta(\psi, \sigma, k) \vee (\delta(\varphi, \sigma, k) \wedge \bigwedge_{c=0}^{k-1}(c, \mathrm{A}\varphi\,\mathcal{U}\,\psi)$
- $\delta(\mathrm{E}\varphi\,\mathcal{U}\,\psi, \sigma, k) = \delta(\psi, \sigma, k) \vee (\delta(\varphi, \sigma, k) \wedge \bigvee_{c=0}^{k-1}(c, \mathrm{E}\varphi\,\mathcal{U}\,\psi)$
- $\delta(\neg\varphi, \sigma, k) = \overline{\delta(\varphi, \sigma, k)}$

Theorem 3 *For every CTL formula φ and a set of directions $\mathcal{D}$ there is an alternating Büchi tree automaton $\mathcal{A}_\varphi$ such that $\mathcal{L}(\mathcal{A}_{\mathcal{D},\varphi})$ is exactly the set of $\mathcal{D}$-branching trees that satisfy φ.*