Automata, Games & Verification

#11

Games

Definition 1. *A game arena is a triple* $\mathcal{A} = (V_0, V_1, E)$*, where*

- V_0 *and* V_1 *are disjoint sets of positions, called the positions of player* 0 *and* 1*,*
- $E \subseteq V \times V$ *for set* $V = V_0 \uplus V_1$ *of game positions,*
- *every position* $p \in V$ *has at least one outgoing edge* $(p, p') \in E$*.*

Definition 2.

- *A reachability game* $\mathcal{G} = (\mathcal{A}, R)$ *consists of a game arena and a winning set of positions* $R \subseteq V$*. Player 0 wins a play* $\pi = p_0 p_1 \ldots$ *if* $p_i \in R$ *for some* $i \in \omega$*, otherwise Player 1 wins.*
- *A Büchi game* $\mathcal{G} = (\mathcal{A}, F)$ *consists of an arena* $\mathcal{A}$ *and a set* $F \subseteq V$*. Player 0 wins a play* π *if* $In(\pi) \cap F \neq \emptyset$*, otherwise Player 1 wins.*
- $\ldots$

Definition 3. *A play is an infinite sequence* $\pi = p_0 p_1 p_2 \ldots \in V^\omega$ *such that* $\forall i \in \omega \, . \, (p_i, p_{i+1}) \in E$.

Definition 4. *A strategy for player* σ *is a function* $f_\sigma : V^* \cdot V_\sigma \to V$ s.t. $(p, p') \in E$ *whenever* $f(u \cdot p) = p'$.

Definition 5. *A play* $\pi = p_0, p_1, \ldots$ *conforms to strategy* f_σ *of player* σ *if* $\forall i \in \omega \, . \,$ *if* $p_i \in V_\sigma$ *then* $p_{i+1} = f_\sigma(p_0, \ldots, p_i)$.

Definition 6.

- *A strategy* f_σ *is* p*-winning for player* σ *and position* p *if all plays that conform to* f_σ *and that start in* p *are won by Player* σ.
- *The winning region for player* σ *is the set of positions*

$$W_\sigma = \{p \in V \mid \text{there is a strategy } f_\sigma \text{ s.t. } f_\sigma \text{ is } p\text{-winning}\}.$$

Definition 7. *A game is determined if* $V = W_0 \cup W_1$.

Definition 8.

- *A memoryless strategy for player* σ *is a function* $f_\sigma : V_\sigma \to V$ *which defines a strategy* $f'_\sigma(u \cdot v) = f(v)$.
- *A game is memoryless determined if for every position some player wins the game with memoryless strategy.*

Solving Reachability Games

Attractor construction:

$$Attr_\sigma^0(X) = \emptyset;$$

$$\begin{aligned} Attr_\sigma^{i+1}(X) = Attr_\sigma^i(X) & \\ \cup & \ \{p \in V_\sigma \mid \exists p' \,.\, (p, p') \in E \wedge p' \in Attr_\sigma^i(X) \cup X\} \\ \cup & \ \{p \in V_{1-\sigma} \mid \forall p' \,.\, (p, p') \in E \Rightarrow p' \in Attr_\sigma^i(X) \cup X\}; \end{aligned}$$

$$Attr_\sigma^+(X) = \bigcup_{i \in \omega} Attr_\sigma^i(X).$$

$$Attr_\sigma(X) = Attr_\sigma^+(X) \cup X$$

Attractor strategy:

- Fix an arbitrary total ordering on V.

- for $p \in V_0$ we define $f_0(q)$:

  -- if $p \in Attr_0^i(R)$ for some smallest $i > 0$,
  choose the minimal $p' \in Attr_0^{i-1}(R) \cup R$ such that $(p, p') \in E$;
  -- otherwise, choose the minimal $p' \in V$ such that $(p, p') \in E$.

Solving Büchi Games

Recurrence construction:

$$Recur_\sigma^0 = F;$$

$$Recur_\sigma^{i+1} = F \cap Attr_\sigma^+(Recur_\sigma^i);$$

$$Recur_\sigma = \bigcap_{i \in \omega} Recur_\sigma^i.$$

Theorem 1. *Reachability and Büchi games are memoryless determined.*