Automata, Games & Verification

Summary #1

- The set of natural numbers $\{0, 1, 2, 3, \ldots\}$ is denoted by ω.
- An alphabet Σ is a finite set of symbols.
- An infinite sequence/string/word is a function from natural numbers to an alphabet:

  $\alpha : \omega \rightarrow \Sigma$

  Notation: $\alpha = \alpha(0)\alpha(1)\alpha(2)\ldots$
- The set of infinite words over alphabet Σ is denoted Σ^ω.
- An ω-language L is a subset of Σ^ω.

Definition 1. *A (nondeterministic) Büchi automaton* $\mathcal{A}$ *over alphabet* Σ *is a tuple* (S, I, T, F)*:*

- S *: a finite set of states;*
- $I \subseteq S$ *: a subset of initial states;*
- $T \subseteq S \times \Sigma \times S$ *: a set of transitions;*
- $F \subseteq S$ *: a subset of accepting states.*

Definition 2. *A run of a nondeterministic Büchi automaton* $\mathcal{A}$ *on an infinite input word* $\alpha = \sigma_0\sigma_1\sigma_2\ldots$ *is an infinite sequence of states* $s_0, s_1, s_2, \ldots$ *such that* $s_0 \in I$ *and for all* $i \in \omega$*,* $(s_i, \sigma_i, s_{i+1}) \in T$*.*

Definition 3. *A Büchi automaton $\mathcal{A}$ is deterministic when*

- I *is a singleton and*
- $\forall \sigma \in \Sigma, \forall s, s_0, s_1 \in S\,.$
  $(s, \sigma, s_0) \in T$ *and* $(s, \sigma, s_1) \in T \;\Rightarrow\; s_0 = s_1$.

Definition 4. *The infinity set of an infinite word $\alpha \in \Sigma^\omega$ is defined as follows*

$$In(\alpha) \;=\; \{\sigma \in \Sigma \,|\, \forall i \exists j\,.\, j \geqslant i \text{ and } \alpha(j) = \sigma\}.$$

Definition 5. [Büchi Acceptance Condition] *A run $r = s_0 s_1 s_2 \ldots$ of a Büchi automaton $\mathcal{A}$ is accepting if*

$$In(r) \cap F \neq \varnothing.$$

Definition 6. *A Büchi automaton $\mathcal{A}$ accepts an infinite word α if there is an accepting run of $\mathcal{A}$ on α.*

Definition 7. *The language recognized by Büchi automaton $\mathcal{A}$ is defined as follows:*

$$\mathcal{L}(\mathcal{A}) = \{\alpha \in \Sigma^\omega \mid \mathcal{A} \text{ accepts } \alpha\}.$$

Definition 8. *An ω-language L is Büchi recognizable if there is a Büchi automaton $\mathcal{A}$ such that $\mathcal{L}(\mathcal{A}) = L$.*

Definition 9. *A Büchi automaton is complete if*

$$\forall s \in S, \sigma \in \Sigma, \exists s' \in S\,.\,(s, \sigma, s') \in T.$$

Theorem 1.

For every Büchi automaton $\mathcal{A}$, there is a complete Büchi automaton $\mathcal{A}'$ such that $\mathcal{L}(\mathcal{A}) = \mathcal{L}(\mathcal{A}')$.

BACKGROUND: The Kleene Theorem

Definition 10. *The regular expressions are defined as follows:*

- *The constants* ϵ *and* $\varnothing$ *are regular expressions.* $\mathcal{L}(\epsilon) = \{\epsilon\}, \mathcal{L}(\varnothing) = \varnothing$.
- *If* $a \in \Sigma$ *is a symbol, then* $\mathbf{a}$ *is a regular expression.* $\mathcal{L}(\mathbf{a}) = \{a\}$.
- *If* E *and* F *are regular expressions, then* $E + F$ *is a regular expression:* $\mathcal{L}(E + F) = \mathcal{L}(E) \cup \mathcal{L}(F)$.
- *If* E *and* F *are regular expressions, then* $E \cdot F$ *is a regular expression:* $\mathcal{L}(E \cdot F) = \{uv \mid u \in \mathcal{L}(E), v \in \mathcal{L}(F)\}$.
- *If* E *is a regular expression, then* E^* *is a regular expression.* $\mathcal{L}(E^*) = \{u_1 u_2 \ldots u_n \mid n \in \omega, u_i \in \mathcal{L}(E) \, \forall \, 0 \leqslant i \leqslant n\}$.

Definition 11. *A language is* *regular* *if it is defined by a regular expression.*

Theorem 2. The Kleene Theorem

A language is regular iff it is recognized by some finite word automaton.