

Automata, Games and Verification: Lecture 9

12 Alternating Automata (Cont'd)

Definition 1 Two nodes $x_1, x_2 \in T$ in a run tree (T, r) are similar if $|x_1| = |x_2|$ and $r(x_1) = r(x_2)$.

Definition 2 A run tree (T, r) is memoryless if for all similar nodes x_1 and x_2 and for all $y \in D^*$ we have that $(x_1 \cdot y \in T \text{ iff } x_2 \cdot y \in T)$ and $r(x_1 \cdot y) = r(x_2 \cdot y)$.

Theorem 1 If an alternating Büchi Automaton \mathcal{A} accepts a word α , then there exists a memoryless accepting run of \mathcal{A} on α .

Proof:

- Let (T, r) be an accepting run tree on α with directions D .
- We define $\gamma : T \rightarrow \omega$ (measures the number of steps since the last visit to F):
 - $\gamma(\epsilon) = 0$
 - $\gamma(n \cdot d) = \begin{cases} \gamma(n) + 1 & \text{if } r(n) \notin F; \\ 0 & \text{otherwise;} \end{cases}$
- We define $\Delta : S \times \omega \rightarrow T$:
 - $\Delta(s, n) = \text{leftmost } y \in T \text{ with } |y| = n, r(y) = s \text{ and } (\forall z \in T, |z| = n \wedge r(z) = s \Rightarrow \gamma(z) \leq \gamma(y))$.
- We define (T', r') :
 - $\epsilon \in T', r'(\epsilon) = r(\epsilon)$;
 - for $n \in T', d \in D$,
 - $n \cdot d \in T' \text{ iff } \Delta(r'(n), |n|) \cdot d \in T$;
 - $r'(n \cdot d) = r(\Delta(r'(n), |n|) \cdot d)$

Claim 1: (T', r') is a run of \mathcal{A} on α .

- $r'(\epsilon) = r(\epsilon) = s_0$
- For $n \in T'$, let $q_n = \Delta(r'(n), |n|)$.
- For every $n \in T', \{r(q_n \cdot d) \mid d \in D, q_n \cdot d \in T\} \models \delta(r(q_n), \alpha(|q_n|))$
and therefore $\{r'(n \cdot d) \mid d \in D, n \cdot d \in T'\} \models \delta(r'(n), \alpha(|n|))$.

Claim 2: If (T, r) is accepting, then so is (T', r') . Proof by contradiction:

- Suppose (T', r') is not accepting, then there is an infinite branch $\pi : n_0, n_1, n_2, \dots \in T'$ and $\exists k \in \omega$ such that $\forall j \geq k : r'(n_j) \notin F$.
- Let $m_i = \Delta(r'(n_i), |n_i|)$ for $i \geq k$.

- **Claim 2.1** : For every $n \in T'$, $\gamma(n) \leq \gamma(\Delta(r'(n), |n|))$. Proof by induction on the length of n :

- for $n = \epsilon$, $\gamma(n) = 0$
- for $n = n' \cdot d$ (where $d \in D$),
 - * if $r(n') \in F$, then $\gamma(n) = 0$
 - * if $r(n') \notin F$, then

$$\begin{aligned}
& \gamma(\Delta(r'(n' \cdot d), |n' \cdot d|)) \\
& \geq \quad (\Delta \text{ definition}) \\
& \quad \gamma(\Delta(r'(n'), |n'|) \cdot d) \\
& = \quad (\gamma \text{ definition}) \\
& \quad 1 + \gamma(\Delta(r'(n'), |n'|)) \\
& \geq \quad (\text{induction hypothesis}) \\
& \quad 1 + \gamma(n') \\
& = \quad (\gamma \text{ definition}) \\
& \quad \gamma(n' \cdot d)
\end{aligned}$$

- We have,

$$\begin{array}{c}
\gamma(n_k) < \gamma(n_{k+1}) < \dots \\
/\wedge \quad \quad \quad / \wedge \\
\gamma(m_k) < \gamma(m_{k+1}) < \dots
\end{array}$$

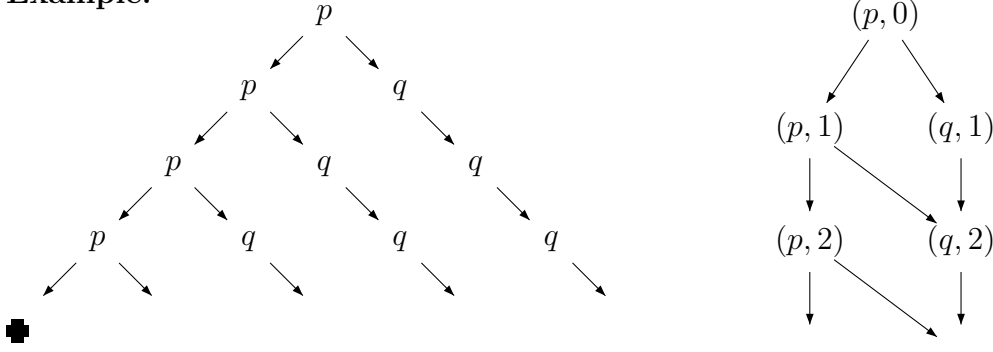
So, for any $k' > k$, $\gamma(m_k) \geq k' - k$.

Since T is finitely branching, there must be a branch with an infinite suffix of non- F labeled positions. This contradicts our assumption that (T, r) is accepting. ■

Definition 3 A run DAG of an alternating Büchi Automaton \mathcal{A} on word α is a DAG (V, E) , where

- $V \subseteq S \times \omega$
- $E \subseteq \bigcup_{i \in \omega} (S \times \{i\}) \times (S \times \{i+1\})$;
- $(s_0, 0) \in V$
- $\forall (s, i) \in V . \exists Y \subseteq S$ s.t.
 $Y \models \delta(s, \alpha(i)), Y \times \{i+1\} \subseteq V$ and $\{(s, i)\} \times (Y \times \{i+1\}) \subseteq E$.

Example:



Notation: Level $((V, E), i) = \{s \in S \mid (s, i) \in V\}$

Definition 4 A run DAG is accepting if every infinite path has infinitely many visits to $F \times \omega$.

Corollary 1 A word α is accepted by an alternating Büchi automaton \mathcal{A} iff \mathcal{A} has an accepting run DAG on α .

Theorem 2 (Miyano and Hayashi, 1984) For every alternating Büchi automaton \mathcal{A} , there exists a nondeterministic Büchi automaton \mathcal{A}' with $\mathcal{L}(\mathcal{A}) = \mathcal{L}(\mathcal{A}')$.

Proof:

- $S' = 2^S \times 2^S$;
- $I' = \{(\{s_0\}, \emptyset)\}$;
- $F' = \{(X, \emptyset) \mid X \subseteq S\}$;
- $T' = \{((X, \emptyset), \sigma, (X', X' - F)) \mid X' \models \bigwedge_{s \in X} \delta(s, \sigma)\} \cup \{((x, W), \sigma, (X', W' \setminus F)) \mid W \neq \emptyset, W' \subseteq X', X' \models \bigwedge_{s \in X} \delta(s, \sigma), W' \models \bigwedge_{s \in W} \delta(s, \sigma)\}$.

$\mathcal{L}(\mathcal{A}') \subseteq \mathcal{L}(\mathcal{A})$:

- Let $\alpha \in L(\mathcal{A}')$ with accepting run

$$r' : (X_0, W_0)(X_1, W_1)(X_2, W_2) \dots$$

where $W_0 = \emptyset, X_0 = \{s_0\}$.

- We construct the run DAG (V, E) for \mathcal{A} on α :
 - $V = \bigcup_{i \in \omega} X_i \times \{i\}$;
 - $E = \bigcup_{i \in \omega} (\bigcup_{x \in X_i \setminus W_i} \{(x, i)\} \times (X_{i+1} \times \{i+1\})) \cup (\bigcup_{x \in W_i} \{(x, i)\} \times ((X_{i+1} \cap (F \cup W_{i+1})) \times \{i+1\}))$.
- (V, E) is an accepting run DAG:
 - $(s_0, 0) \in V$;
 - for $(x, i) \in V$:
 - * if $x \in X_i \setminus W_i, X_{i+1} \models \delta(x, \alpha(i))$;

- * if $x \in W_i$, $X_{i+1} \cap (F \cup W_{i+1}) \models \delta(x, \alpha(i))$.
- Every path through the run DAG visits F infinitely often (otherwise $W_i = \emptyset$ only for finitely many i).

$\mathcal{L}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{A}')$:

- Let $\alpha \in L(\mathcal{A}')$ and (V, E) an accepting run DAG of \mathcal{A} on α .
- We construct a run

$$r' : (X_0, W_0)(X_1, W_1)(X_2, W_2) \dots$$

on \mathcal{A}' as follows:

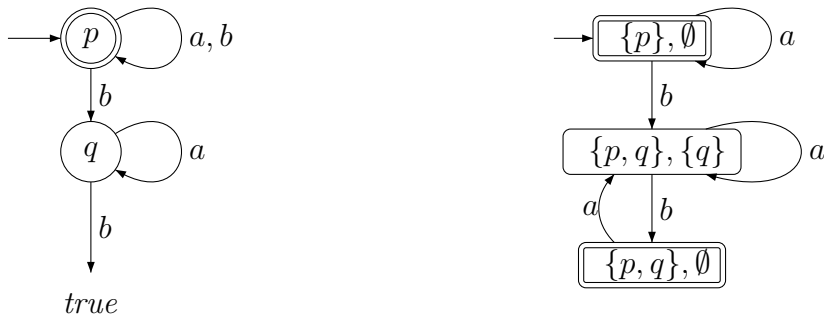
- $X_0 = \{s_0\}$, $W_0 = \emptyset$;
- for $i > 0$, $X_i = \text{Level}((V, E), i)$
 - * if $W_i = \emptyset$ then $W_{i+1} = X_{i+1} \setminus F$,
 - * otherwise,

$$W_{i+1} := \{y' \in S \setminus F \mid \exists (y, i) \in V, ((y, i), (y', i+1)) \in E, y \in W_i\}.$$

- r' is an accepting run:
 - starts with $(\{s_0\}, \emptyset)$
 - obeys T' :
 - * for $x \in X_i \setminus W_i$, $X_{i+1} \models \delta(x, \alpha(i))$;
 - * for $x \in W_i$, $X_{i+1} \cap (F \cup W_{i+1}) \models \delta(x, \alpha(i))$.
 - r' is accepting (otherwise there exists a path in (V, E) that is not accepting).

■

Example: We translate the following *universal* automaton (all branchings are conjunctions) into an equivalent nondeterministic automaton:



■

Corollary 2 *A language is ω -regular iff it is recognizable by an alternating Büchi automaton.*

Proof:

Translation from nondeterministic Büchi automaton $(S, \{s_0\}, T, F)$ to alternating Büchi automaton (S, s_0, δ, F) with

$$\bullet \delta(s, \sigma) = \bigvee_{s' \in pr_3(T \cap \{s\} \times \{\sigma\} \times S)} s' \quad \text{for all } s \in S$$

■

Comment: Acceptance of a word α by an alternating Büchi automaton can also be characterized as a game:

- Positions of Player 0: $V_0 = S \times \omega$;
- Positions of Player 1: $V_1 = 2^S \times \omega$;
- Edges: $\{((s, i), (X, i)) \mid X \models \delta(s, \alpha(i))\}$
 $\cup \{((X, i), (s, i + 1)) \mid s \in X\}$

Player 0 wins a play iff $F \times \omega$ is visited infinitely often.

The word α is accepted iff Player 0 has a strategy to win the game from position $(s_0, 0)$.

End Comment