Bernd Finkbeiner Date: May 3, 2011

Automata, Games and Verification: Lecture 4

5 Complementation of Nondeterministic Büchi Automata (Cont'd)

Lemma 1 *If $\mathcal{A}$ does not accept α, then there exists an odd ranking for G.*

Proof:

- We define $f(\langle s,l\rangle) = 2i$ if $\langle s,l\rangle$ is endangered in G_{2i} and
- $f(\langle s,l\rangle) = 2i+1$ if $\langle s,l\rangle$ is safe in G_{2i+1}.
- f is a ranking:
  - by Lemma 2 from Lecture 3, G_j is empty for $j > 2\cdot|S|$. Hence, $f : V \to \{0,\ldots,2\cdot|S|\}$.
  - if $\langle s',l'\rangle$ is a successor of $\langle s,l\rangle$, then $f(\langle s',l'\rangle) \leq f(\langle s,l\rangle)$
    - Let $j := f(\langle s,l\rangle)$.
    - Case j is even: vertex $\langle s,l\rangle$ is endangered in G_j; hence either $\langle s',l'\rangle$ is not in G_j, and therefore $f(\langle s,l\rangle) < j$; or $\langle s',l'\rangle$ is in G_j and endangered; hence, $f(\langle s,l\rangle) = j$.
    - Case j is odd: vertex $\langle s,l\rangle$ is safe in G_j; hence either $\langle s',l'\rangle$ is not in G_j, and therefore $f(\langle s,l\rangle) < j$; or $\langle s',l'\rangle$ is in G_j and safe; hence, $f(\langle s,l\rangle) = j$.
  - f is an odd ranking:
    - For every path $\langle s_0,l_0\rangle, \langle s_1,l_1\rangle, \langle s_2,l_2\rangle, \ldots$ in G there exists an $i \geq 0$ such that for all $j \geq 0$, $f(\langle s_{i+j},l_{i+j}\rangle) = f(\langle s_i,l_i\rangle)$.
    - Suppose that $k := f(\langle s_i,l_i\rangle)$ is even. Thus, $\langle s_i,l_i\rangle$ is endangered in G_k.
    - Since $f(\langle s_{i+j},l_{i+j}\rangle) = k$ for all $j \geq 0$, all $\langle s_{i+j},l_{i+j}\rangle$ are in G_k.
    - This contradicts that $\langle s_i,l_i\rangle$ is endangered in G_k.

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Theorem 1 *For each Büchi automaton $\mathcal{A}$ there exists a Büchi automaton $\mathcal{A}'$ such that $\mathcal{L}(\mathcal{A}') = \Sigma^\omega \setminus \mathcal{L}(\mathcal{A})$.*

Helpful definitions:

- A *level ranking* is a function $g : S \to \{0,\ldots,2\cdot|S|\} \cup \{\bot\}$ such that if $g(s)$ is odd, then $s \notin F$.

- Let $\mathcal{R}$ be the set of all level rankings.
- A level ranking g' *covers* a level ranking g if, for all $s, s' \in S$, if $g(s) \geq 0$ and $(s, \sigma, s') \in T$, then $0 \leq g'(s') \leq g(s)$.

Proof:

We define $\mathcal{A}' = (S', I', T', F')$ with

- $S' = \mathcal{R} \times 2^S$;
- $I' = \{\langle g_0, \emptyset \rangle \mid g_0 \in \mathcal{R}, g_0(s) = \bot \text{ iff } s \notin I\}$;
- $T = \{(\langle g, \emptyset \rangle, \sigma, \langle g', P' \rangle) \mid g' \text{ covers } g, \text{ and } P' = \{s' \in S \mid g'(s') \text{ is even }\}\}$
  $\cup\ \{(\langle g, P \rangle, \sigma, \langle g', P' \rangle) \mid P \neq \emptyset, g' \text{ covers } g, \text{ and}$
  $P' = \{s' \in S \mid (s, \sigma, s') \in T, s \in P, g'(s') \text{ is even }\}\}$;
- $F = \mathcal{R} \times \{\emptyset\}$.

(Intuition: $\mathcal{A}'$ guesses the level rankings for the run DAG. The P component tracks the states whose corresponding vertices in the run DAG have even ranks. Paths that traverse such vertices should eventually reach a vertex with odd rank. The acceptance condition ensures that all paths visit a vertex with odd rank infinitely often.)

$\mathcal{L}(\mathcal{A}') \subseteq \Sigma^\omega \setminus \mathcal{L}(\mathcal{A})$:

- Let $\alpha \in \mathcal{L}(\mathcal{A}')$ and let $r' = (g_0, P_0), (g_1, P_1), \ldots$ be an accepting run of $\mathcal{A}'$ on α.
- Let $G = (V, E)$ be the run DAG of $\mathcal{A}$ on α.
- The function $f : \langle s, l \rangle \mapsto g_l(s), s \in S_l, l \in \omega$ is a ranking for G:
  - if $g_i(s)$ is odd then $s \notin F$;
  - for all $(\langle s, l \rangle, \langle s', l+1 \rangle) \in E$, $g_{l+1}(s') \leq g_l(s)$.
- f is an odd ranking:
  - Assume otherwise. Then there exists a path $\langle s_0, l_0 \rangle, \langle s_1, l_1 \rangle, \langle s_2, l_2 \rangle, \ldots$ in G such that for infinitely many $i \in \omega$, $f(\langle s_i, l_i \rangle)$ is even.
  - Hence, there exists an index $j \in \omega$, such that $f(\langle s_j, l_j \rangle)$ is even and, for all $k \geq 0$, $f(\langle s_{j+k}, l_{j+k} \rangle) = f(\langle s_j, l_j \rangle)$.
  - Since r' is accepting, $P_{j'} = \emptyset$ for infinitely many j'. Let j' be the smallest such index $\geq j$.
  - $P_{j'+1+k} \neq \emptyset$ for all $k \geq 0$.
  - Contradiction.
- Since there exists an odd ranking, $\alpha \notin \mathcal{L}(\mathcal{A})$.

$\Sigma^\omega \setminus \mathcal{L}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{A}')$:

- Let $\alpha \in \Sigma^\omega \setminus \mathcal{L}(\mathcal{A})$ and let $G = (V, E)$ be the run DAG of $\mathcal{A}$ on α.

- There exists an odd ranking f on G.
- There is a run $r' = (g_0, P_0), (g_1, P_1), \ldots$ of $\mathcal{A}'$ on α, where
  $g_l(s) = \begin{cases} f(\langle s, l \rangle) & \text{if } s \in S_l; \\ \bot & \text{otherwise;} \end{cases}$
  $P_0 = \emptyset$,
  $P_{l+1} = \begin{cases} \{s \in S \mid g_{l+1}(s) \text{ is even }\} & \text{if } P_l = \emptyset, \\ \{s' \in S \mid \exists s \in S_l \cap P_l \ . \ (\langle s, l \rangle, \langle s', l+1 \rangle) \in E, g_{l+1}(s') \text{ is even}\} & \text{otherwise.} \end{cases}$
- r' is accepting. (Assume there is an index i such that $P_j \neq \emptyset$ for all $j \geq i$. Then there exists a path in G that visits an even rank infinitely often.)
- Hence, $\alpha \in \mathcal{L}(\mathcal{A}')$.

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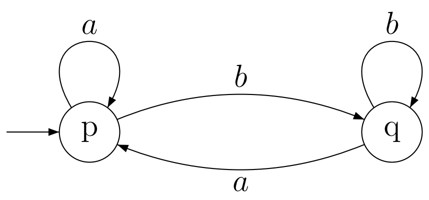
6 Muller Automata

Definition 1 *A* (nondeterministic) Muller automaton $\mathcal{A}$ *over alphabet* Σ *is a tuple* $(S, I, T, \mathcal{F})$*:*

- S, I, T *: defined as before*
- $\mathcal{F} \subseteq 2^S$ *: set of* accepting subsets*, called the* table*.*

Definition 2 *A run* r *of a Muller automaton is* accepting *iff* $In(r) \in F$

Example:



- for $\mathcal{F} = \{\{q\}\}$: $\mathcal{L}(\mathcal{A}) = (a \cup b)^* b^\omega$
- for $\mathcal{F} = \{\{q\}, \{p, q\}\}$: $\mathcal{L}(\mathcal{A}) = (a^* b)^\omega$

✚

Theorem 2 *For every (deterministic) Büchi automaton $\mathcal{A}$, there is (deterministic) Muller automaton $\mathcal{A}'$, such that $\mathcal{L}(\mathcal{A}) = \mathcal{L}(\mathcal{A}')$.*

Proof:

$S' = S$, $I' = I$, $T' = T$
$\mathcal{F}' = \{Q \subseteq S \mid Q \cap F \neq \emptyset\}$ ■

Theorem 3 *For every nondeterministic Muller automaton $\mathcal{A}$ there is a nondeterministic Büchi automaton $\mathcal{A}'$ such that $\mathcal{L}(\mathcal{A}) = \mathcal{L}(\mathcal{A}')$.*

Proof:

- $\mathcal{F}' = \{F_1, \ldots, F_n\}$
- $S' = S \cup \bigcup_{i=1}^{n} \{i\} \times F_i \times 2^{F_i}$
- $I' = I$
- $T' = T$
  $\cup\ \{(s, \sigma, (i, s', \emptyset)) | 1 \leq i \leq n, (s, \sigma, s') \in T, s' \in F_i\}$
  $\cup\ \{((i, s, R), \sigma, (i', s', R')) \mid 1 \leq i \leq n, s, s' \in F_i, R, R' \subseteq F_i,$
  $(s, \sigma, s') \in T, R' = R \cup \{s\}$ if $R \neq F_i$ and $R' = \emptyset$ if $R = F_i\}$
- $F' = \bigcup_{i=1}^{n} \{i\} \times F_i \times \{F_i\}$

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