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Automata, Games and Verification: Lecture 3

4 Deterministic Büchi Automata

Theorem 1 *The language $(a+b)^* b^\omega$ is not recognizable by a deterministic Büchi automaton.*

Proof:

- Assume that L is recognized by the deterministic Büchi automaton $\mathcal{A}$.
- Since $b^\omega \in L$, there is a run
  $r_0 = s_{0,0}s_{0,1}s_{0,2}, \ldots$
  with $s_{0,n_0} \in F$ for some $n_0 \in \omega$.
- Similarly, $b^{n_0}ab^\omega \in L$ and there must be a run
  $r_1 = s_{0,0}s_{0,1}s_{0,2} \ldots s_{0,n_0}s_1s_{1,0}s_{1,1}s_{1,2} \ldots$
  with $s_{1,n_1} \in F$
- Repeating this argument, there is a word $b^{n_0}ab^{n_1}ab^{n_2}a \ldots$ accepted by $\mathcal{A}$.
- This contradicts $L = \mathcal{L}(\mathcal{A})$.

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Definition 1 (Substrings) *Let $\alpha \in \Sigma^\omega$. For $n, m \in \omega, n \leq m$ we define*

$$\alpha(n,m) = \alpha(n)\alpha(n+1)\ldots\alpha(m)\ .$$

Definition 2 (Limit) *For $W \subseteq \Sigma^*$:*

$$\overrightarrow{W} = \{\alpha \in \Sigma^\omega \mid \textit{there exist infinitely many } n \in \omega \textit{ s.t. } \alpha(0,n) \in W\}\ .$$

Theorem 2 *An ω-language $L \subseteq \Sigma^\omega$ is recognizable by a deterministic Büchi automaton iff there is a regular language $W \subseteq \Sigma^*$ s.t. $L = \overrightarrow{W}$.*

Proof:

Let L be the language of a deterministic Büchi automaton $\mathcal{A}$; let W be the regular language of $\mathcal{A}$ as a deterministic finite-word automaton. We show that $L = \overrightarrow{W}$.

$$\begin{aligned} & \alpha \in L \\ \text{iff}\quad & \text{for the unique run } r \text{ of } \mathcal{A} \text{ on } \alpha,\ In(r) \cap F \neq \emptyset \\ \text{iff}\quad & \alpha(0,n) \in W \text{ for infinitely many } n \in \omega \\ \text{iff}\quad & \alpha \in \overrightarrow{W}. \end{aligned}$$

■

Theorem 3 *For any deterministic Büchi automaton $\mathcal{A}$, there exists a Büchi automaton $\mathcal{A}'$ such that $\mathcal{L}(\mathcal{A}') = \Sigma^\omega \setminus \mathcal{L}(\mathcal{A})$.*

Proof:

We construct $\mathcal{A}'$ as follows:

- $S' = (S \times \{0\}) \cup ((S \setminus F) \times \{1\})$.
- $I' = I \times \{0\}$.
- $T' = \{((s,0), \sigma, (s',0)) \mid (s,\sigma,s') \in T\}$
  $\cup \{((s,0), \sigma, (s',1)) \mid (s,\sigma,s') \in T, s' \in S - F\}$
  $\cup \{((s,1), \sigma, (s,1)) \mid (s,\sigma,s') \in T, s' \in S - F\}$.
- $F' = (S - F) \times \{1\}$.

$\mathcal{L}(\mathcal{A}') \subseteq \Sigma^\omega - \mathcal{L}(\mathcal{A})$:

- For $\alpha \in \mathcal{L}(\mathcal{A}')$ we have an accepting run

  $$r' : (s_0, 0)(s_1, 0) \ldots (s_j, 0)(s'_0, 1)(s'_1, 1) \ldots$$

  on $\mathcal{A}'$.
- Hence,

  $$r : s_0 s_1 s_2 \ldots s_j s'_0 s'_1 \ldots$$

  is the unique run on α in $\mathcal{A}$.
- Since $s'_0, s'_1, \ldots \in S \setminus F$, $\mathit{In}(r) \subseteq S \setminus F$. Hence, r is not accepting and $\alpha \in \Sigma^\omega - \mathcal{L}(\mathcal{A})$

$\mathcal{L}(\mathcal{A}') \supseteq \Sigma^\omega - \mathcal{L}(\mathcal{A})$:

- We assume $\alpha \notin \mathcal{L}(\mathcal{A})$. Since $\mathcal{A}$ is deterministic and complete there exists a run

  $$r : s_0 s_1 s_2 \ldots$$

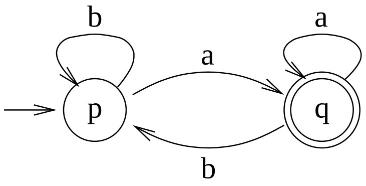
  for α on $\mathcal{A}$, but $\mathit{In}(r) \cap F = \emptyset$.
- Thus there exists a $k \in \omega$ such that $s_j \notin F$ for $j > k$.
- This gives us the run

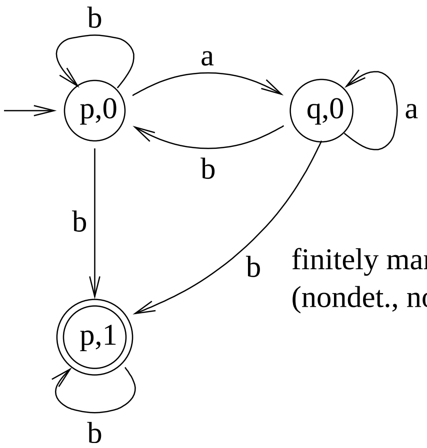
  $$r' : (s_0, 0)(s_1, 0) \ldots (s_k, 0)(s_{k+1}, 1)(s_{k+2}, 1) \ldots$$

  for α on $\mathcal{A}'$ with the property $\mathit{In}(r') \subseteq ((S - F) \times \{1\}) = F'$.
- Hence, r' is accepting and $\alpha \in \mathcal{L}(\mathcal{A}')$.

$\blacksquare$

Example:





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5 Complementation of Nondeterministic Büchi Automata

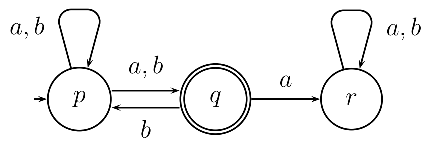
Reference: The following construction for the complementation of nondeterministic Büchi automata is taken from: Orna Kupferman and Moshe Y. Vardi, Weak alternating automata are not that weak. *ACM Trans. Comput. Logic* 2, 3 (Jul. 2001), 408-429.

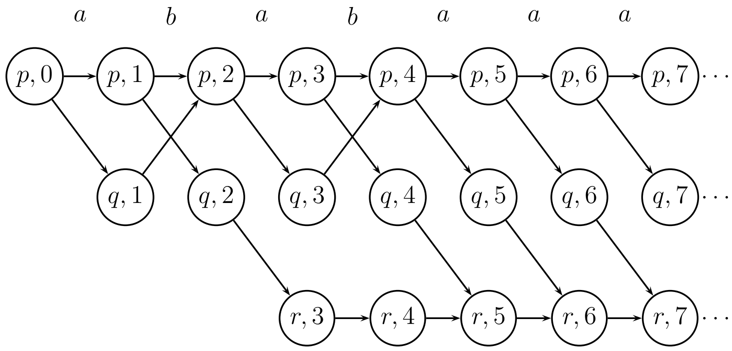
Definition 3 *Let $\mathcal{A} = (S, I, T, F)$ be a nondeterministic Büchi automaton. The* run DAG *of $\mathcal{A}$ on a word $\alpha \in \Sigma^\omega$ is the directed acyclic graph $G = (V, E)$ where*

- $V = \bigcup_{l \geq 0} (S_l \times \{l\})$ *where* $S_0 = I$ *and* $S_{l+1} = \bigcup_{s \in S_l, (s, \alpha(l), s') \in T} \{s'\}$
- $E = \{(\langle s, l \rangle, \langle s', l+1 \rangle) \mid l \geq 0, (s, \alpha(l), s') \in T\}$

A path in a run DAG is accepting iff it visits $F \times \mathbb{N}$ infinitely often. The automaton accepts α if some path is accepting.

Example:





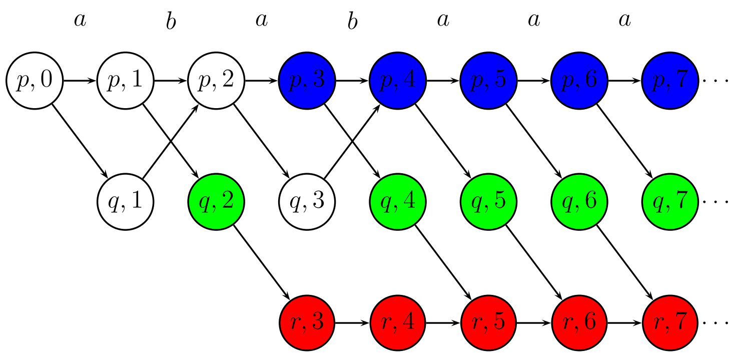
♣

Definition 4 *A* ranking *for G is a function $f : V \to \{0, \ldots, 2 \cdot |S|\}$ such that*

- *for all $\langle s, l\rangle \in V$, if $f(\langle s, l\rangle)$ is odd then $s \notin F$;*
- *for all $(\langle s, l\rangle, \langle s', l'\rangle) \in E$, $f(\langle s', l'\rangle) \leq f(\langle s, l\rangle)$.*

A ranking is *odd* iff for all paths $\langle s_0, l_0\rangle, \langle s_1, l_1\rangle, \langle s_2, l_2\rangle, \ldots$ in G, there is a $i \geq 0$ such that $f(\langle s_i, l_i\rangle)$ is odd and, for all $j \geq 0$, $f(\langle s_{i+j}, l_{i+j}\rangle) = f(\langle s_i, l_i\rangle)$.

Example:



rank 1 — rank 2 — rank 3 — rank 4

♣

Lemma 1 *If there exists an odd ranking for G, then $\mathcal{A}$ does not accept α.*

Proof:

- In an odd ranking, every path eventually gets trapped in a some odd rank.
- If $f(\langle s, l \rangle)$ is odd, then $s \notin F$.
- Hence, every path visits F only finitely often.

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Let G' be a subgraph of G. We call a vertex $\langle s, l \rangle$

- *safe* in G' if for all vertices $\langle s', l' \rangle$ reachable from $\langle s, l \rangle$, $s' \notin F$, and
- *endangered* in G' if only finitely many vertices are reachable.

We define an infinite sequence $G_0 \supseteq G_1 \supseteq G_2 \supseteq \ldots$ of DAGs inductively as follows:

- $G_0 = G$
- $G_{2i+1} = G_{2i} \smallsetminus \{\langle s, l \rangle \mid \langle s, l \rangle \text{ is endangered in } G_{2i}\}$
- $G_{2i+2} = G_{2i+1} \smallsetminus \{\langle s, l \rangle \mid \langle s, l \rangle \text{ is safe in } G_{2i+1}\}$.

Lemma 2 *If $\mathcal{A}$ does not accept α, then the following holds: For every $i \geq 0$ there exists an l_i such that for all $j \geq l_i$ at most $|S| - i$ vertices of the form $\langle \_, j \rangle$ are in G_{2i}.*

Proof:

Proof by induction on i:

- $i = 0$: In G, for every l, there are at most $|S|$ vertices of the form $\langle \_, l \rangle$.
- $i \to i + 1$:
  - Case G_{2i} is finite: then $G_{2(i+1)}$ is empty.
  - Case G_{2i} is infinite:
    - There must exist a safe vertex $\langle s, l \rangle$ in G_{2i+1}. (Otherwise, we can construct a path in G with infinitely many visits to F).
    - We choose $l_{i+1} = l$.
    - We prove that for all $j \geq l$, there are at most $|S| - (i + 1)$ vertices of the form $\langle \_, j \rangle$ in G_{2i+2}.
      - Since $\langle s, l \rangle \in G_{2i+1}$, it is not endangered in G_{2i}.
      - Hence, there are infinitely many vertices reachable from $\langle s, l \rangle$ in G_{2i}.
      - By König's Lemma, there exists an infinite path $p = \langle s, l \rangle, \langle s_1, l + 1 \rangle, \langle s, l + 2 \rangle, \ldots$ in G_{2i}.
      - No vertex on p is endangered (there is an infinite path). Therefore, p is in G_{2i+1}.
      - All vertices on p are safe ($\langle s, l \rangle$ is safe) in G_{2i+1}. Therefore, none of the vertices on p are in G_{2i+2}.
      - Hence, for all $j \geq l$, the number of vertices of the form $\langle \_, l \rangle$ in G_{2i+2} is strictly smaller than their number in G_{2i}.

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