Bernd Finkbeiner Date: April 12, 2011

Automata, Games and Verification: Lecture 1

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0 Course Organization

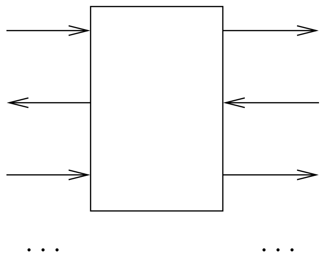
- Vertiefungsvorlesung (6 CP)
- react.cs.uni-sb.de
- Bernd Finkbeiner: E1.3/506, office hours Wednesdays 3-4
- Rüdiger Ehlers: E1.3/532
- Andrey Kupriyanov: E1.3/508
- Tutorial: Thursdays 10-12, Seminarraum 0.01, building E 2 1 (bioinformatics)
- Your final grade will depend 100% on the final exam.
- Every week, assignments will be given out and solutions presented the following week. Credit points will be given to students who present correct solutions to problems during the tutorial. Each problem will be assigned in advance to a group of two members (single person groups are allowed). The solutions will not be graded, since only participation is required. Problems will be distributed fairly (on a rotating scheme), but only those with enough points (max 1 unexcused no-show) may write the final exam.
- challenge problems: not assigned to any group; will earn you bonus points, take you out of the rotation once
- Literature:
  Erich Grädel et al: Automata, Logics, and Infinite Games (available online)
  Khoussainov/Nerode: Automata Theory and its Applications
  Lecture notes (online after lecture)
  Summary slides (online after lecture)

1 Motivation

We distinguish

- Transformational programs

- Reactive systems



  - nonterminating behavior
  - interaction (program vs. environment)

1.1 Problem 1: Verification

Example: Mutual execution with program TURN

$$\text{local } t\text{: boolean where initially } t = 0$$

$$P_0 :: \left[\begin{array}{l} \text{loop forever do} \\ \qquad \left[\begin{array}{ll} 00: & \text{noncritical;} \\ 01: & \text{await } t = 0; \\ 10: & \text{critical;} \\ 11: & t := 1; \end{array}\right] \end{array}\right] \;||\; P_1 :: \left[\begin{array}{l} \text{loop forever do} \\ \qquad \left[\begin{array}{ll} 00: & \text{noncritical;} \\ 01: & \text{await } t = 1; \\ 10: & \text{critical;} \\ 11: & t := 0; \end{array}\right] \end{array}\right]$$

TURN is a finite-state program with 32 states, which can be encoded as bit vectors $(b_1, b_2, b_3, b_4, b_5)$, with (b_1, b_2) for the location of P_0, (b_3, b_4) for the location of P_1, and b_5 for t.

Behavior: infinite sequence of states
Specification: set of correct behaviors

Example: specifications:

- Mutual execution: it is never the case that P_0 and P_1 are in their critical sections, i.e. the states 10100 and 10101 do not occur

- Accessibility: whenever P_i is in location 01 it will eventually reach location 10

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The Verification Problem: Given a program P and a specification φ, decide whether P satisfies φ.

Underlying concept: Automata over infinite words (more generally: objects)

Solution:

1. Construct automaton that accepts all sequences that are
   - possible in P and
   - violate φ.
2. Check automaton for emptiness.

1.2 Problem 2: Synthesis

Example: Mutual execution by arbiter

$$\text{local } r_0, r_1, g_0, g_1\text{: boolean where initially } r_1 = r_2 = g_1 = g_2 = 0$$

$$P_0 :: \left[\begin{array}{l} \text{loop forever do} \\ \quad \left[\begin{array}{ll} 00: & r_0 := 1; \\ 01: & \text{await } g_0 = 1; \\ 10: & r_0 := 0; \\ 11: & \text{critical;} \end{array}\right] \end{array}\right] \; || \; P_1 :: \left[\begin{array}{l} \text{loop forever do} \\ \quad \left[\begin{array}{ll} 00: & r_1 := 1; \\ 01: & \text{await } g_1 = 1; \\ 10: & r_1 := 0; \\ 11: & \text{critical;} \end{array}\right] \end{array}\right] \; || \; \text{Arbiter:: } \textbf{?}$$

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The Synthesis Problem: Given a specification φ, decide if *there exists* a program P that satisfies φ. If yes: construct such a program.

Underlying concept: Infinite games.
Play of the game = infinite sequence of states.
Player "system" wins the game if sequence satisfies φ for all possible behaviors of player "environment".

Solution:

1. Decide whether player "system" has a winning strategy.
2. If yes, construct a program that implements that strategy.

1.3 History

1960 – 1970 Fundamental results about ω-automata and games. Motivation: Logical decision problems, circuit design.

- **J. Richard Büchi** (1924-1984)
  Swiss logician and mathematician; Ph.D. at ETH, then Purdue University, Lafayette, Indiana. Inventor of Büchi automata. Great influence on theoretical computer science, combinatorics, graph theory.
- **Robert McNaughton**
  taught philosophy; then switched to computer science in 1950s; emeritus at Harvard; McNaughton's theorem: each recognizable set of infinite words can be recognized by a deterministic ω-automaton.
- **Michael Rabin** (*1931, Breslau)
  won Turing award together with Dana Scott for inventing nondeterministic machines; proved that second order theory of n successors is decidable; determinacy of parity games.

Since 1980: Revival of the theory in the setting of temporal logics

Motivation today:

- industrial use (especially finite-state verification "model checking")
- decidability of many problems with infinite structures
- bridge between logic and computer science

2 Büchi Automata

2.1 Basic Definitions

- The *set of natural numbers* $\{0, 1, 2, 3, \ldots\}$ is denoted by ω.
- An *alphabet* Σ is a finite set of symbols.
- An *infinite sequence/string/word* is a function from natural numbers to an alphabet:
  $\alpha : \omega \to \Sigma$
  An infinite word is composed of its letters, so that in particular $\alpha = \alpha(0)\alpha(1)\alpha(2)\ldots$
- The *set of infinite words over alphabet* Σ is denoted Σ^ω (finite words: Σ^*).
- An ω*-language* L is a subset of Σ^ω.

Example:

- $\emptyset$ is the *empty* ω-language.

- $\{a^\omega\} = \{aaaa\ldots\}$;
- $\{ba^\omega, aba^\omega, aaba^\omega, \ldots\}$.

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Definition 1 *A* nondeterministic Büchi automaton $\mathcal{A}$ *over alphabet* Σ *is a tuple* (S, I, T, F)*:*

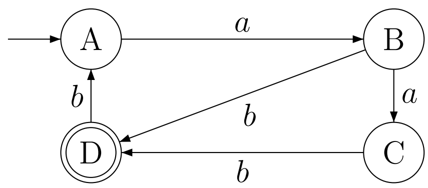
- S *: a finite set of* states
- $I \subseteq S$ *: a subset of* initial states
- $T \subseteq S \times \Sigma \times S$ *: a set of* transitions
- $F \subseteq S$ *: a subset of* accepting states

Now we define how a Büchi automaton uses an infinite word as input. Notice that we do not refer to acceptance in this definition.

Definition 2 *A* run *of a nondeterministic Büchi automaton* $\mathcal{A}$ *on an infinite input word* $\alpha = \sigma_0\sigma_1\sigma_2\ldots$ *is an infinite sequence of states* $s_0, s_1, s_2, \ldots$ *such that the following hold:*

- $s_0 \in I$
- *for all* $i \in \omega$*,* $(s_i, \sigma_i, s_{i+1}) \in T$

Example:



In the automaton shown the set of states are $S = \{A, B, C, D\}$, the initial set of states are $I = \{A\}$ (indicated with pointing arrow with no source), the transitions $T = \{(A, a, B), (B, a, C), (C, b, D), (D, b, A)\}$ are the remaining arrows in the diagram, and the set of accepting states is $F = \{D\}$ (double-lined state circle).

On input $aabbaabb\ldots$ the Büchi automaton shown has only the run:

$ABCDABCDABCD\ldots$ ✚

Determinism is a property of machines that can only react in a unique way to their input. The following definition makes this clear for Büchi automata.

Definition 3 *A Büchi automaton $\mathcal{A}$ is* deterministic *when T is a partial function (with respect to the next input letter and the current state):*
$\forall \sigma \in \Sigma, \forall s, s_0, s_1 \in S \,.\, (s, \sigma, s_0) \in T$ *and* $(s, \sigma, s_1) \in T \;\Rightarrow\; s_0 = s_1$
and I is singleton.

(By Büchi automaton we usually mean nondeterministic Büchi automaton.)

Definition 4 *The* infinity set *of an infinite word $\alpha \in \Sigma^\omega$ is the set $In(\alpha) = \{\sigma \in \Sigma \mid \forall i \exists j \,.\, j \geq i$ and $\alpha(j) = \sigma\}$*

Definition 5

- *A Büchi automaton $\mathcal{A}$* accepts *an infinite word α if:*
  - *there is a run $r = s_0 s_1 s_2 \ldots$ of α on $\mathcal{A}$*
  - *r is* accepting*: $In(r) \cap F \neq \emptyset$*
- *The* language recognized by Büchi automaton *$\mathcal{A}$ is defined as follows:*
  $\mathcal{L}(\mathcal{A}) = \{\alpha \in \Sigma^\omega \mid \mathcal{A}$ *accepts* $\alpha\}$

Example: Automaton $\mathcal{A}$ from previous example. $\mathcal{L}(\mathcal{A}) = \{aabbaabbaabb\ldots\}$. ✚

Comment: A deterministic Büchi automaton $\mathcal{A} = (S, I, T, F)$ defines a partial function[1] from Σ^ω to a set of runs $R \subseteq S^\omega$. **End Comment**

Definition 6 *An ω-language L is Büchi recognizable if there is a Büchi automaton $\mathcal{A}$ such that $\mathcal{L}(\mathcal{A}) = L$.*

Example: The singleton ω-language $L = \{\sigma\}$ with $\sigma = abaabaaabaaaab\ldots$ is not Büchi recognizable. (Note that all finite languages of finite words are NFA-recognizable. Analog result does not hold for Büchi-automata)

Proof:

- Suppose there is a Büchi automaton $\mathcal{A}$ with $\mathcal{L}(\mathcal{A}) = L$.
- Let $r = s_0 s_1 \ldots$ be an accepting run on σ.
- Since F is finite, there exists $k, k' \in \omega$ with $k < k'$ and $s_k = s_{k'} \in F$.
- $r' = r_0 \ldots r_{k'-1}(r_k \ldots r_{k'-1})^\omega$ is an accepting run on $\sigma' = \sigma(0) \ldots \sigma(k'-1)(\sigma(k) \ldots \sigma(k'-1))^\omega$.
- Hence, $\sigma' \in \mathcal{L}(\mathcal{A})$. Contradiction.

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Definition 7 *A Büchi automaton is* complete *if its transition relation contains a function:*
$\forall s \in S, \sigma \in \Sigma \,.\, \exists s' \in S \,.\, (s, \sigma, s') \in T$

[1] A *partial function* is a function that is not defined on all of the elements of its domain.

Theorem 1 *For every Büchi automaton $\mathcal{A}$, there is a complete Büchi automaton $\mathcal{A}'$ such that $\mathcal{L}(\mathcal{A}) = \mathcal{L}(\mathcal{A}')$.*

Proof:

We define $\mathcal{A}'$ in terms of the components S, I, T, F of $\mathcal{A}$:

$$S' = S \cup \{f\} \qquad f \text{ new}$$

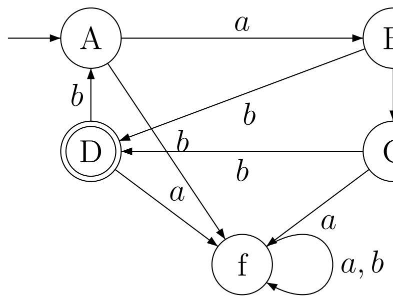
$$I' = I$$

$$T' = T \cup \{(s, \sigma, f) \mid \nexists s' . (s, \sigma, s') \in T\} \cup \{(f, \sigma, f) \mid \sigma \in \Sigma\}$$

$$F' = F$$

The runs of $\mathcal{A}'$ are a superset of those of $\mathcal{A}$ since we have added states and transitions. Furthermore, on any infinite input word α the accepting runs of $\mathcal{A}$ and $\mathcal{A}'$ correspond, because any run that reaches f stays in f, and since $f \notin F'$, such a run is not accepting. ■

Example: Completing the Büchi automaton from a previous example we obtain the following automaton:



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Unless we specify otherwise, we will only consider complete automata when we prove results.

Comment: A complete deterministic Büchi automaton $\mathcal{A} = (S, I, T, F)$ may be viewed as a total function[2] from Σ^ω to S^ω. A complete (possibly nondeterministic) Büchi automaton can produce at least one run for every Σ^ω input word.
End Comment

[2] A total function, in contrast to a partial one, is defined on its entire domain.