Automata, Games & Verification

Summary #3

Don’t forget to register through HISPOS.
Deadline: TOMORROW May 16

Today at 4:00pm in SR 014: First meeting of the seminar.
Games in Verification and Synthesis
Piotr Danilewski: *Algorithms for solving parity games*

Complementation

Theorem 1. *For any deterministic Büchi automaton $\mathcal{A}$, there exists a Büchi automaton $\mathcal{A}'$ such that $\mathcal{L}(\mathcal{A}') = \Sigma^\omega \setminus \mathcal{L}(\mathcal{A})$.*

Proof: We construct $\mathcal{A}'$ as follows:

- $S' = (S \times \{0\}) \cup ((S \setminus F) \times \{1\})$.
- $I' = I \times \{0\}$.
- $$T' = \{((s,0),\sigma,(s',0)) \mid (s,\sigma,s') \in T\} \\ \cup \ \{((s,0),\sigma,(s',1)) \mid (s,\sigma,s') \in T\} \\ \cup \ \{((s,1),\sigma,(s,1)) \mid (s,\sigma,s') \in T, s' \in S - F\}.$$
- $F' = (S - F) \times \{1\}$.

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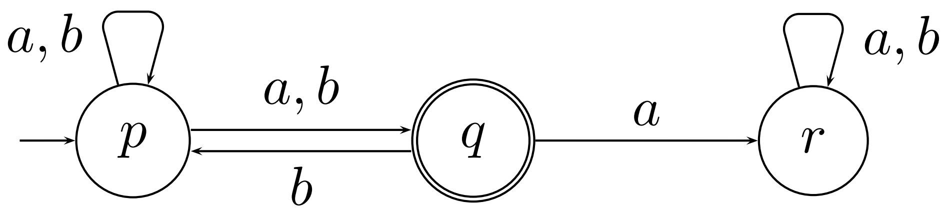
Definition 1. *Let* $\mathcal{A} = (S, I, T, F)$ *be a nondeterministic Büchi automaton. The run DAG of* $\mathcal{A}$ *on a word* $\alpha \in \Sigma^\omega$ *is the directed acyclic graph* $G = (V, E)$ *where*

- $V = \bigcup_{l \geqslant 0} (S_l \times \{l\})$ *where* $S_0 = I$ *and* $S_{l+1} = \bigcup_{s \in S_l, (s, \alpha(l), s') \in T} \{s'\}$

- $E = \{(\langle s, l \rangle, \langle s', l+1 \rangle) \mid l \geqslant 0, (s, \alpha(l), s') \in T\}$

A path in a run DAG is accepting iff it visits F infinitely often.
The automaton accepts α if some path is accepting.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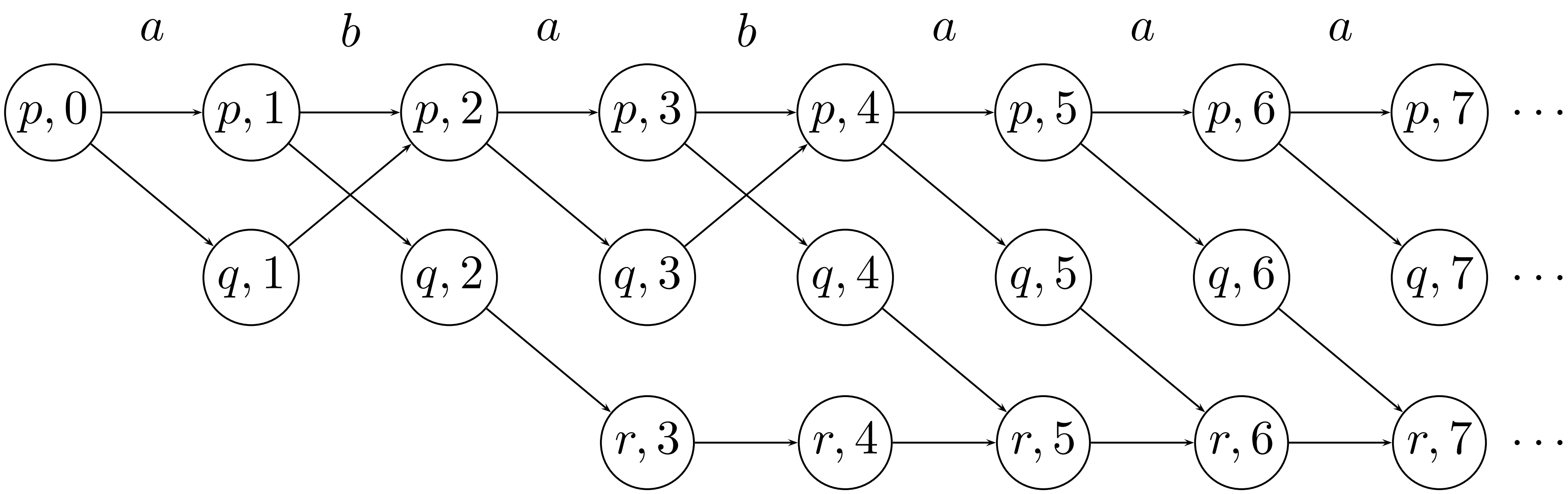
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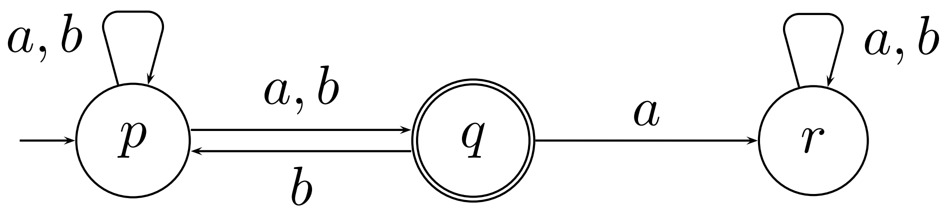
Definition 2. *A ranking for G is a function $f : V \to \{0, \ldots, 2 \cdot |S|\}$ such that*

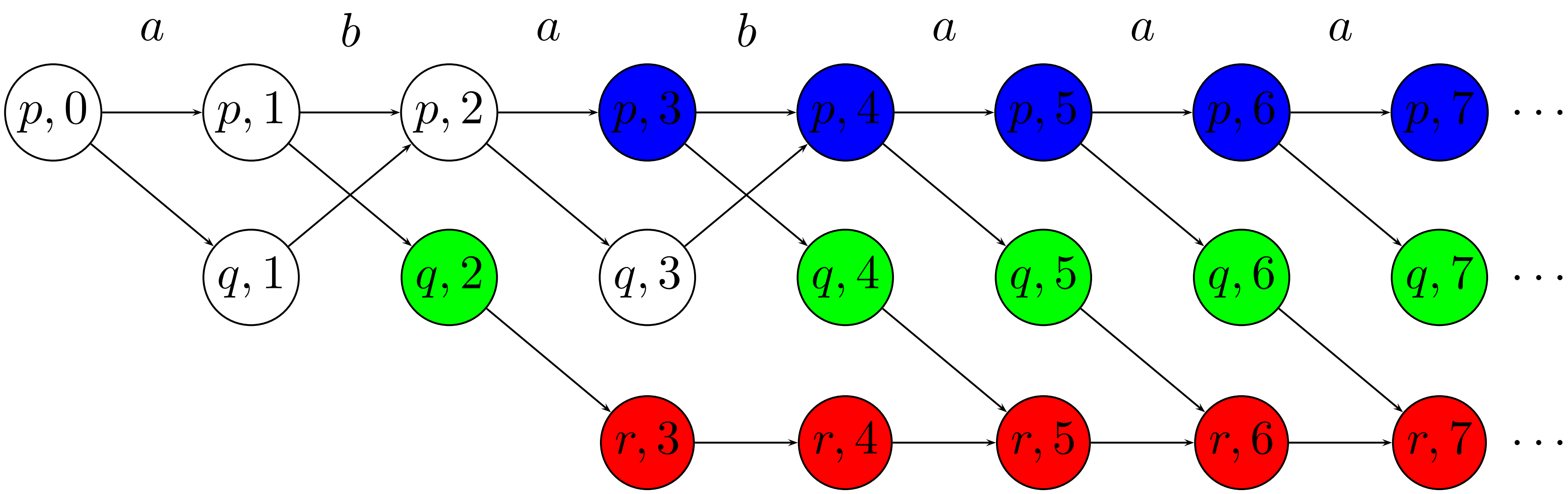
- *for all $\langle s, l \rangle \in V$, if $f(\langle s, l \rangle)$ is odd then $s \notin F$;*
- *for all $(\langle s, l \rangle, \langle s', l' \rangle) \in E$, $f(\langle s', l' \rangle) \leqslant f(\langle s, l \rangle)$.*

A ranking is odd iff for all paths $\langle s_0, l_0 \rangle, \langle s_1, l_1 \rangle, \langle s_2, l_2 \rangle, \ldots$ in G, there is a $i \geqslant 0$ such that $f(\langle s_i, l_i \rangle)$ is odd and, for all $j \geqslant 0$, $f(\langle s_{i+j}, l_{i+j} \rangle) = f(\langle s_i, l_i \rangle)$.

Lemma 1.

If there exists an odd ranking for G, then $\mathcal{A}$ does not accept α.





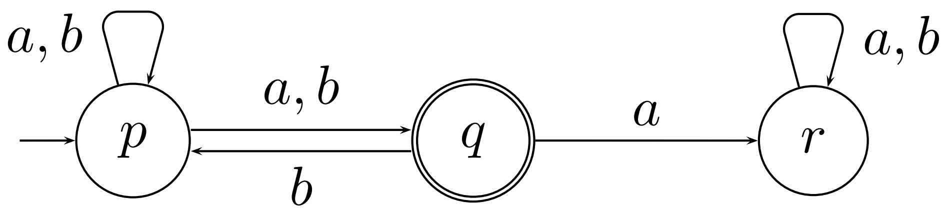
rank 1 — rank 2 — rank 3 — rank 4

Let G' be a subgraph of G. We call a vertex $\langle s, l \rangle$

- safe in G' if for all vertices $\langle s', l' \rangle$ reachable from $\langle s, l \rangle$, $s' \notin F$, and
- endangered in G' if only finitely many vertices are reachable.

We define an infinite sequence $G_0 \supseteq G_1 \supseteq G_2 \supseteq \ldots$ of DAGs inductively as follows:

- $G_0 = G$
- $G_{2i+1} = G_{2i} \setminus \{\langle s, l \rangle \mid \langle s, l \rangle$ is endangered in $G_{2i}\}$
- $G_{2i+2} = G_{2i+1} \setminus \{\langle s, l \rangle \mid \langle s, l \rangle$ is safe in $G_{2i}\}$.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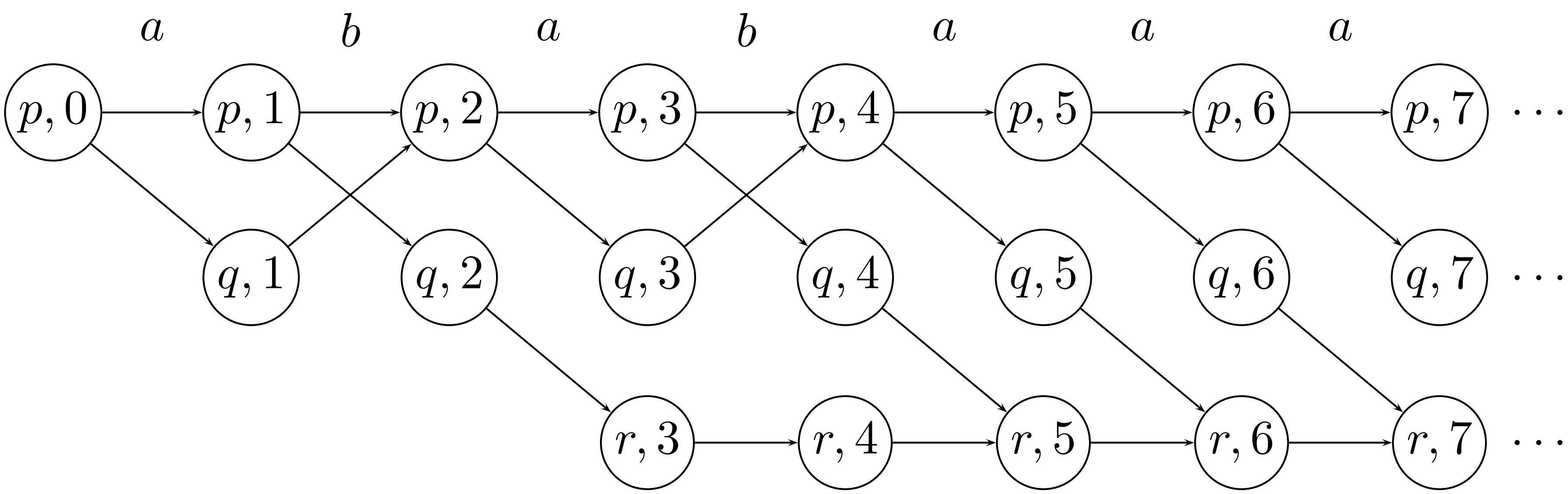
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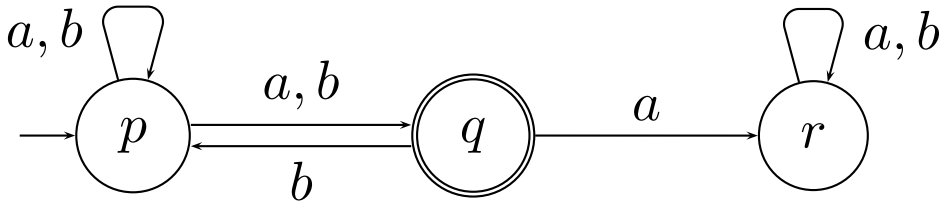
$$G = G_0 = G_1$$



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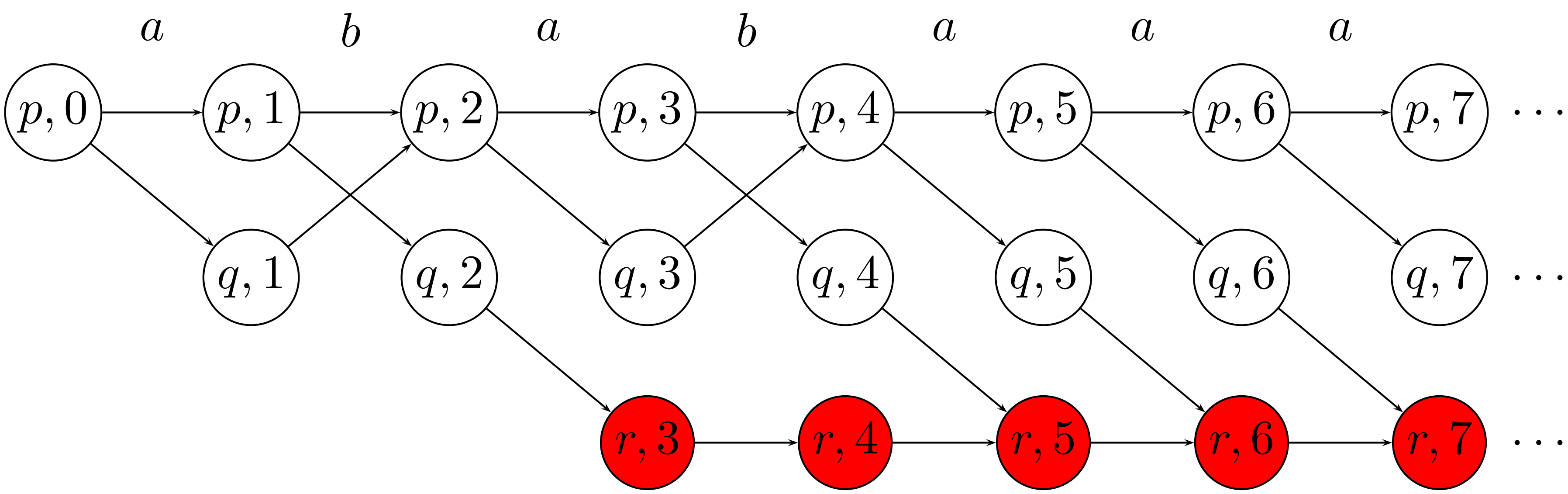

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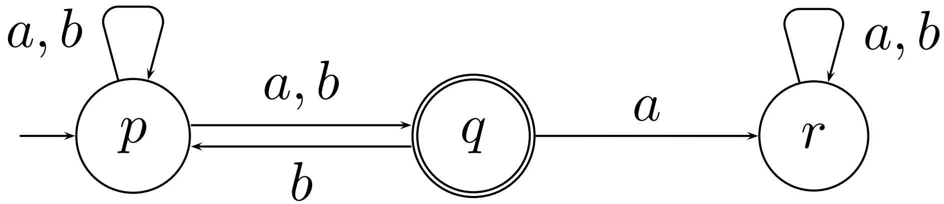
G_1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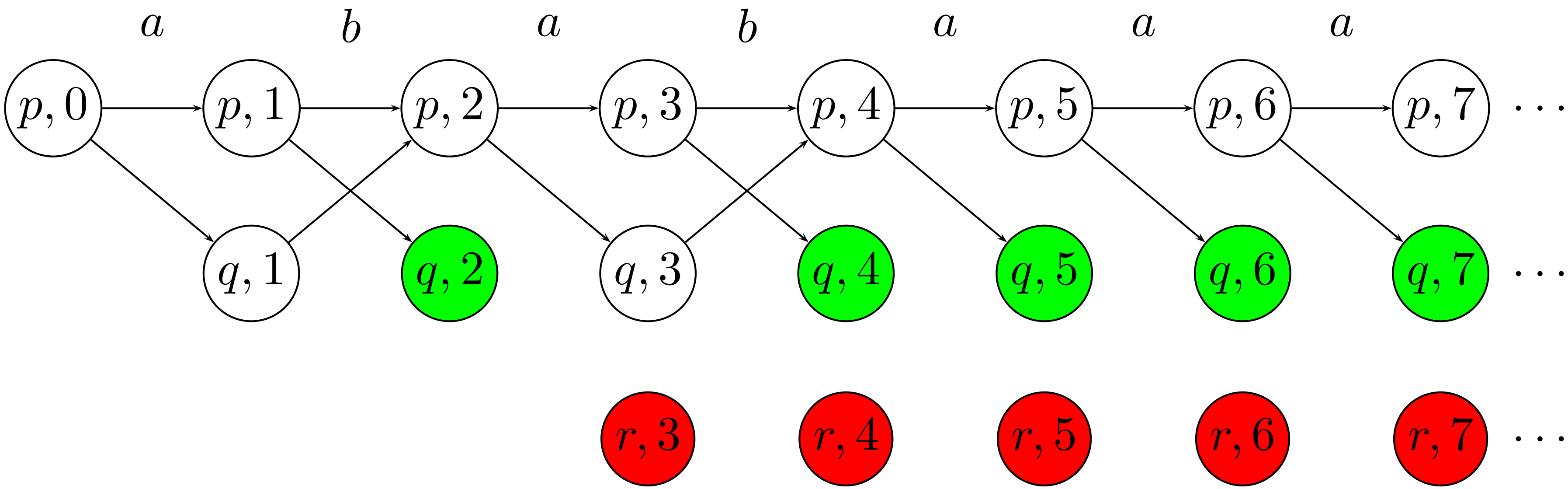
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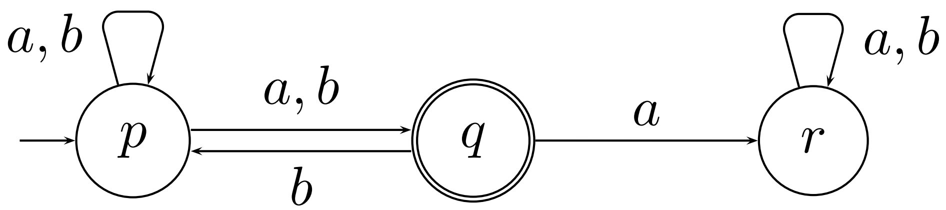
G_2



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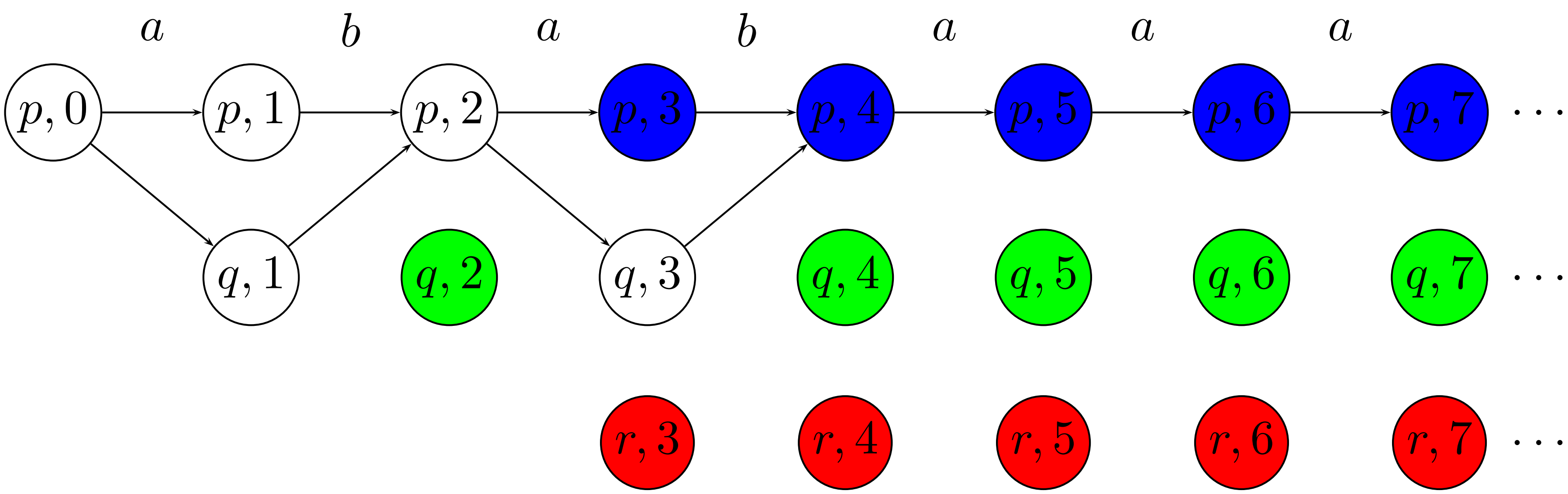

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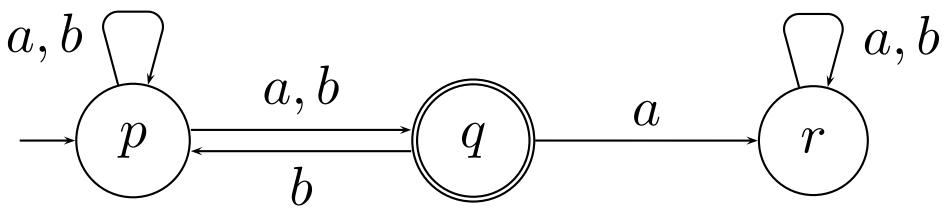
G_3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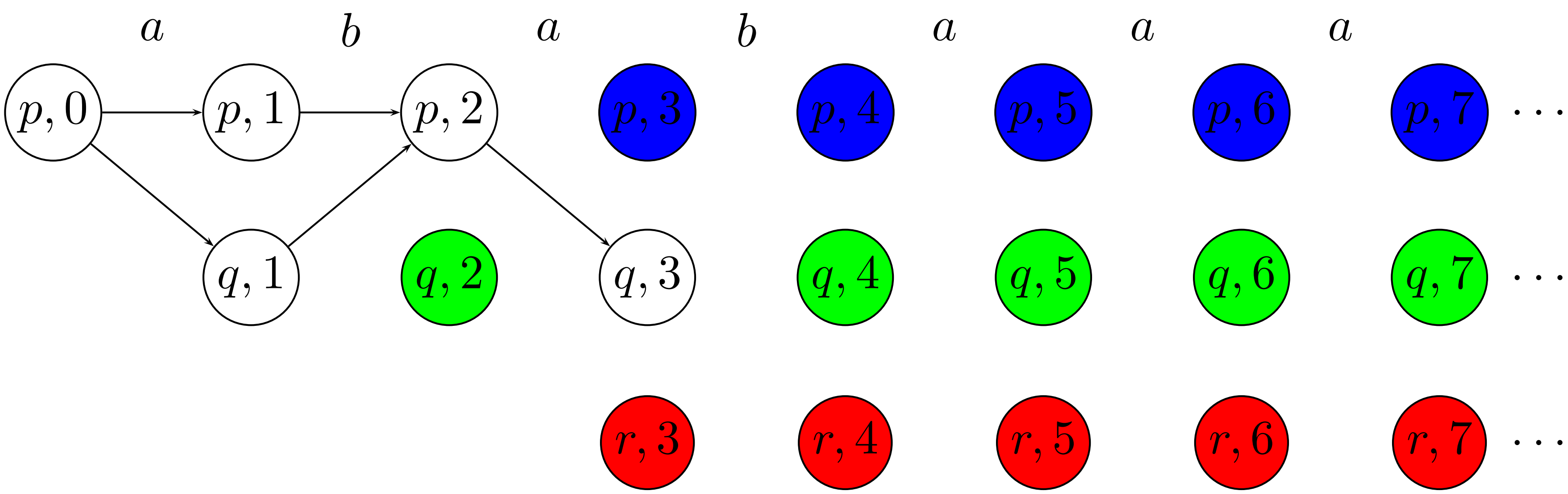
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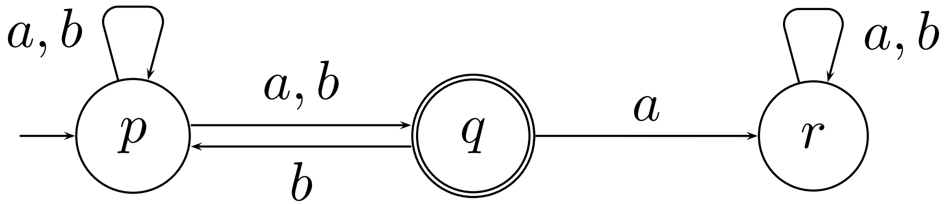
G_4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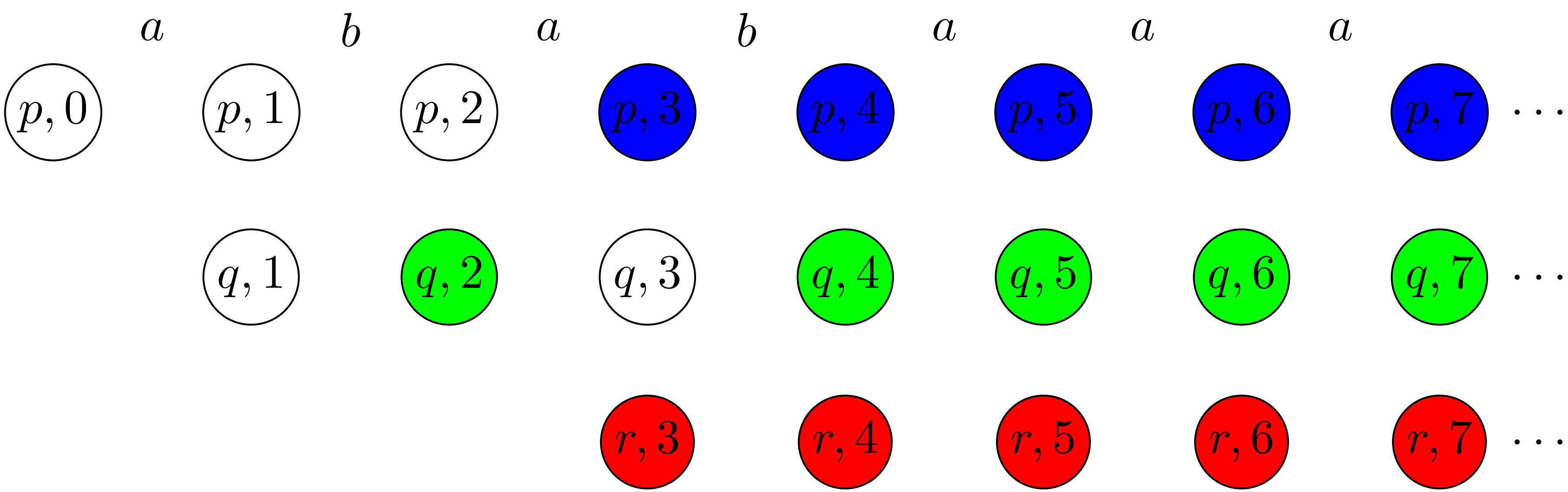
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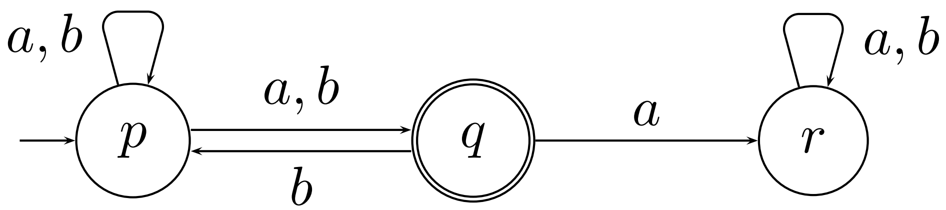
G_5

Lemma 2.

If $\mathcal{A}$ does not accept α, then the following holds:
For every $i \geqslant 0$ there exists an l_i such that
for all $j \geqslant l_i$ at most $|S| - i$ vertices of the form $\langle \_, j \rangle$ are in G_{2i}.

Lemma 3.

If $\mathcal{A}$ does not accept α, then there exists an odd ranking for G.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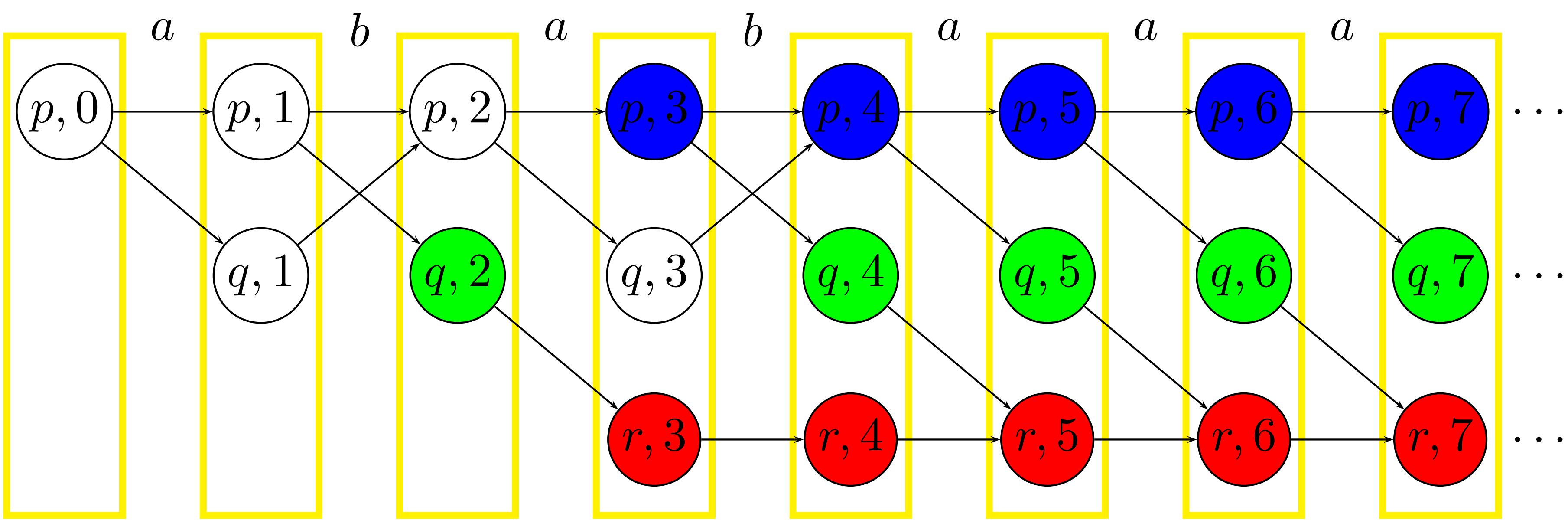
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