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Automata, Games and Verification: Lecture 7

9 Quantified Propositional Temporal Logic (QPTL)

Syntax: LTL formula | $\varphi \wedge \varphi$ | $\neg\varphi$ | $\exists p.\ \varphi$

Semantics:

$$\alpha, i \models \exists q.\varphi \quad \text{iff} \quad \begin{array}{l}\text{there is an } \alpha' \text{ with} \\ \alpha'(j) \cap (AP \smallsetminus \{q\}) = \alpha(j) \cap (AP \smallsetminus \{q\}) \text{ for all } j \in \omega, \\ \text{s.t. } \alpha', i \models \varphi.\end{array}$$

Example: $L = (\emptyset\emptyset)^*\{p\}^\omega$ is QPTL-definable:

$\exists q.\ (q \wedge \Box(q \leftrightarrow \neg q) \wedge \Box(p \rightarrow \bigcirc p) \wedge \Box(\bigcirc p \leftrightarrow p \vee q))$ ✚

Theorem 1 *For every Büchi automaton $\mathcal{A}$ over $\Sigma = 2^{AP}$ there exists a QPTL formula φ such that models$(\varphi) = \mathcal{L}(\mathcal{A})$.*

Proof:

Let $S = \{s_1, s_2, \ldots, s_n\}$ and $AP' = AP \cup \{at_{s_1}, \ldots, at_{s_n}\}$.

$$\begin{aligned}\varphi \;:=\; \exists at_{s_1}, \ldots, at_{s_n} \quad . \quad & \bigvee_{s \in I} at_s \\ \wedge \quad & \Box \left(\bigvee_{(s_i, A, s_j) \in T} at_{s_i} \wedge \bigcirc at_{s_j} \wedge \left(\bigwedge_{p \in A} p \right) \wedge \left(\bigwedge_{p \in AP \smallsetminus A} \neg p \right) \right) \\ \wedge \quad & \Box \left(\bigvee_{i=1}^{n} \bigwedge_{j \neq i} \neg (at_{s_i} \wedge at_{s_j}) \right) \\ \wedge \quad & \Box\Diamond \bigvee_{s_i \in F} at_{s_i}\end{aligned}$$

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10 Monadic Second-Order Theory of One Successor (S1S)

Syntax:

- first-order variable set $V_1 = \{x, y, \ldots\}$

- second-order variable set $V_2 = \{X, Y, \ldots\}$
- Terms t:

$$t ::= 0 \mid x \mid S(t)$$

- Formulas φ:

$$\varphi ::= t \in X \mid t_1 = t_2 \mid \neg\varphi \mid \varphi_0 \vee \varphi_1 \mid \exists x.\varphi \mid \exists X.\varphi$$

Abbreviations:

- $\forall X.\ \varphi \quad := \ \neg\exists X.\ \neg\varphi;$
- $x \notin Y \ := \ \neg(x \in Y);$
- $x \neq y \ := \ \neg(x = y).$

Semantics:

- first-order valuation $\sigma_1 : V_1 \to \omega$
- second-order valuation $\sigma_2 : V_2 \to 2^\omega$

Semantics of terms:

- $[0]_{\sigma_1} = 0$
- $[x]_{\sigma_1} = \sigma_1(x)$
- $[S(t)_{\sigma_1}] = [t]_{\sigma_1} + 1$

Semantics of formulas:

- $\sigma_1, \sigma_2 \models t \in X$ iff $[t]_{\sigma_1} \in \sigma_2(X)$
- $\sigma_1, \sigma_2 \models t_1 = t_2$ iff $[t_1]_{\sigma_1} = [t_2]_{\sigma_1}$
- $\sigma_1, \sigma_2 \models \neg\psi$ iff $\sigma_1, \sigma_2 \not\models \psi$
- $\sigma_1, \sigma_2 \models \psi_0 \vee \psi_1$ iff $\sigma_1, \sigma_2 \models \psi_0$ or $\sigma_1, \sigma_2 \models \psi_1$
- $\sigma_1, \sigma_2 \models \exists x.\ \varphi$ iff there is an $a \in \omega$ s.t.

$$\sigma_1'(y) = \begin{cases} \sigma_1(y) \text{ if } y \neq x \\ a \text{ otherwise} \end{cases}$$

  and $\sigma_1', \sigma_2 \models \varphi$.

- $\sigma_1, \sigma_2 \models \exists X.\ \varphi$ iff there is an $A \subseteq \omega$ s.t.

$$\sigma_2'(Y) = \begin{cases} \sigma_2(Y) \text{ if } Y \neq X \\ A \text{ otherwise} \end{cases}$$

  and $\sigma_1, \sigma_2' \models \varphi$

Example:

$$
\begin{aligned}
X \subseteq Y &:\equiv \forall z.\ (z \in X \rightarrow z \in Y);\\
X = Y &:\equiv X \subseteq Y \wedge Y \subseteq X;\\
\mathit{Suff}(X) &:\equiv \forall y.\ (y \in X \rightarrow S(y) \in X);\\
x \leq y &:\equiv \forall Z.\ (x \in Z \wedge \mathit{Suff}(Z)) \rightarrow y \in Z;\\
\mathit{Fin}(X) &:\equiv \exists Y.\ ((X \subseteq Y \wedge \exists z - \ z \notin Y \wedge \forall z.\ (z \notin Y \rightarrow S(z)) \notin Y);
\end{aligned}
$$

✚

Definition 1 *For a S1S formula φ, models(φ) $= \{\alpha_{\sigma_1,\sigma_2} \in (2^{V_1 \cup V_2})^\omega \mid \sigma_1, \sigma_2 \models \varphi\}$, where $x \in \alpha(j)$ iff $j = \sigma_1(x)$, and $X \in \alpha(j)$ iff $j \in \sigma_2(X)$.*

Definition 2 *A language L is LTL/QPTL/S1S-definable if there is a LTL/QPTL/S1S formula φ with models(φ) $= L$.*

Theorem 2 *Every QPTL-definable language is S1S-definable.*

Proof:

For every QPTL-formula φ over AP and every S1S-term t over $V_1 = \emptyset$, we define a S1S formula $T(\varphi, t)$ over $V_1 = \emptyset$, $V_2 = AP$, such that, for all $\alpha \in (2^{AP})^\omega$,

$$\alpha, [t]_{\sigma_1} \models_{\text{QPTL}} \varphi \qquad \text{iff} \qquad \sigma_1, \sigma_2 \models_{\text{S1S}} T(\varphi, t),$$

where $\sigma_2 : P \mapsto \{i \in \omega \mid P \in \alpha(i)\}$.

- $T(P, t) = t \in P$, for $P \in AP$;
- $T(\neg\varphi, t) = \neg T(\varphi, t)$;
- $T(\varphi \vee \psi, t) = T(\varphi, t) \vee T(\psi, t)$
- $T(\bigcirc\varphi, t) = T(\varphi, S(t))$
- $T(\varphi\, \mathcal{U}\, \psi, t) = \exists y.(y \geq t \wedge T(\psi, y) \wedge \neg\exists z.(x \leq z < y \wedge T(\neg\varphi, z)))$
- $T(\exists P\ \varphi, t) = \exists P.\ T(\varphi, t)$.

models(φ) = models($T(\varphi, 0)$). ■

Theorem 3 *Every S1S-definable language is Büchi-recognizable.*

Proof:

Let φ be a S1S-formula.

1. Rewrite φ into normal form
   $\varphi ::= 0 \in X \mid x \in Y \mid x = 0 \mid x = y \mid x = S(y) \mid$
   $\neg\varphi \mid \varphi \vee \psi \mid \exists x.\ \varphi \mid \exists X.\ \varphi$.
   using the following rewrite rules:

$$
\begin{aligned}
S(t) \in X &\mapsto \exists y.\ y = S(t) \wedge y \in X\\
S(t) = S(t') &\mapsto t = t'\\
S(t) = x &\mapsto x = S(t)\\
t = S(S(t')) &\mapsto \exists y.\ y = S(t') \wedge t = S(y)
\end{aligned}
$$

2. Rename bound variables to obtain unique variables.

   Example:

$$\exists x.(S(S(y)) = x \wedge \exists x\ (S(x) \in X_0))$$

   is rewritten to

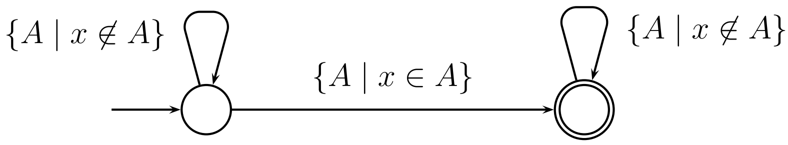
$$\exists x_0.\ \exists x_1.x_0 = S(x_1) \wedge x_1 = S(y) \wedge \exists x_2 \exists x_3.x_3 = S(x_2) \wedge x_3 \in X_0$$

3. Construct Büchi automaton:

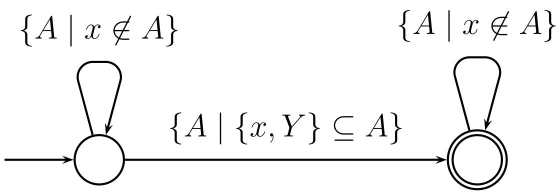
   Base cases:

   - $0 \in X$:

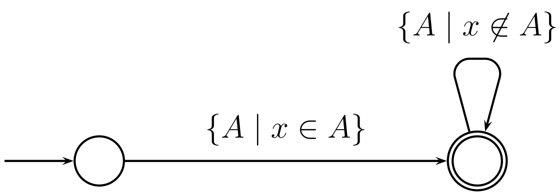
   For every $x \in V_1$, intersect with $\mathcal{A}_x$:



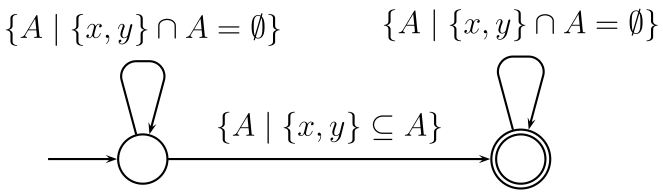
   - $x \in Y$:



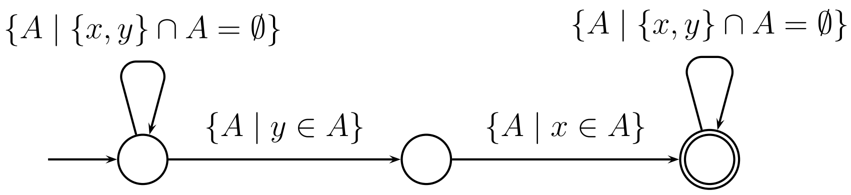
   - $x = 0$:



- $x = y$:



- $x = S(y)$:



Inductive step:

- $\varphi \vee \psi$: language union,
- $\neg\varphi$: complement (and intersection with all $\mathcal{A}_x$),
- $\exists x.\ \varphi$: projection (and intersection with $\mathcal{A}_x$),
- $\exists X.\ \varphi$: projection.

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