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Date: May 29, 2008

Automata, Games and Verification: Lecture 5

7 McNaughton's Theorem

Theorem 1 (McNaughton's Theorem (1966)) *Every Büchi recognizable language is recognizable by a deterministic Muller automaton.*

Definition 1 *A Büchi automaton* (S, I, T, F) *is called* semi-deterministic *if* $S = N \uplus D$ *is a partition of* S, $F \subseteq D$ *and* $(D, \{d\}, T, F)$ *is deterministic for every* $d \in D$.

Lemma 1 *For every Büchi automaton* $\mathcal{A}$ *there exists a semi-deterministic Büchi automaton* $\mathcal{A}'$ *with* $\mathcal{L}(\mathcal{A}) = \mathcal{L}(\mathcal{A}')$.

Proof:

Given $\mathcal{A} = (S, I, T, F)$, we construct $\mathcal{A}' = (S', I', T', F')$:

- $S' = 2^S \uplus 2^S \times 2^S$;
- $I' = \{I\}$;
- $T' = \{(L, \sigma, L') \mid L' = pr_3(T \cap L \times \{\sigma\} \times S)\}$;
  $\cup\{(L, \sigma, (\{s'\}, \emptyset)) \mid \exists s \in L.\, (s, \sigma, s') \in T\}$
  $\cup\{((L_1, L_2), \sigma, (L_1', L_2')) \mid L_1 \neq L_2$
  $L_1' = pr_3(T \cap L_1 \times \{\sigma\} \times S)$,
  $L_2' = pr_3(T \cap L_1 \times \{\sigma\} \times F) \cup pr_3(T \cap L_2 \times \{\sigma\} \times S)\}$
  $\cup\{((L, L), \sigma, (L_1', L_2')) \mid L_1' = pr_3(T \cap L_1 \times \{\sigma\} \times S)$,
  $L_2' = pr_3(T \cap L_1 \times \{\sigma\} \times F)\}$
- $F' = \{(L, L) \mid L \neq \emptyset)\}$

$\mathcal{L}(\mathcal{A}') \subseteq \mathcal{L}(\mathcal{A})$:

- Let $\alpha \in \mathcal{L}(\mathcal{A}')$.
- Let $r' = P_0, P_1, \ldots, P_n, (L_0, L_0'), (L_1, L_1'), \ldots$ be an accepting run of $\mathcal{A}'$ on α.
- For every $s \in L_0$ there is a run prefix of $\mathcal{A}$ on $\alpha(0, n)$, $p_0, p_1, \ldots, p_n, s$ such that $p_j \in P_j$ and
- Let $i_0, i_1, \ldots$ be an infinite sequence of indices such that $i_0 = 0$, $L_{i_j} = L_{i_j}'$, $L_{i_j} \neq \emptyset$ for all $j \in \omega$.
- For every $j > 1$, and every $s' \in L_{i_j}$ there exists a state $s \in L_{i_{j-1}}$ and a sequence $s = s_{i_{j-1}}, s_{i_{j-1}+1}, \ldots, s_{i_j} = s'$ such that $(s_k, \alpha(k), s_{k+1}) \in T$ for all $k \in \{i_{j-1}, \ldots, i_{i_j-1}\}$ and $s_k \in F$ for some $k \in \{i_{j-1}+1, \ldots, i_{i_j}\}$.
  Let $predecessor(s', i_j) := s$,
  $run(s', i_0) = p_0, p_1, \ldots, p_n, s'$ where $L_0 = \{s'\}$, and
  $run(s', i_j) = s_{i_{j-1}+1}, s_{i_{j-1}+2}, \ldots, s_{i_j}$, for $j > 0$.

- Consider the following $\left(\bigcup_{j\in\omega} L_{i_j} \times \{j\}\right)$-labeled tree:
  - the root is labeled with $(s, 0)$, where $L_0 = \{s\}$, and
  - the parent of each node labeled with (s', j) is labeled with $(\mathit{predecessor}(s', i_j), j - 1)$.
- The tree is infinite and finite-branching, and, hence, by König's Lemma, has an infinite branch $(s_{i_0}, i_0), (s_{i_1}, i_1), \ldots$, corresponding to an accepting run of $\mathcal{A}$:

$$\mathit{run}(s_{i_0}, i_0) \cdot \mathit{run}(s_{i_1}, i_1) \cdot \mathit{run}(s_{i_2}, i_2) \cdot \ldots$$

$\underline{\mathcal{L}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{A}')}$:

- Let $\alpha \in \mathcal{L}(\mathcal{A})$.
- Let $r = s_0, s_1, \ldots$ be an accepting run of $\mathcal{A}$ on α.
- Let i be an index s.t. $s_i \in F$ and for all $j \geq i$ there exists a $k > j$, such that

$$\{s \in S \mid s_i \rightarrow^{\alpha(i,k)} s\} = \{s \in S \mid s_j \rightarrow^{\alpha(j,k)} s\}.$$

  This index exists:
  - "$\supseteq$" holds for all i, because there is a path through s_j.
  - Assume that for all i, there is a $j \geq i$ s.t for all $k > j$ "$\supsetneq$" holds. Then there exists an i' s.t. $\{s \in S \mid s_{i'} \rightarrow^{\alpha(i',k)} s\} = \emptyset$ for all $k > i'$. Contradiction.
- We define a run r' of $\mathcal{A}'$:

$$r' = P_0, \ldots, P_{i-1}, (\{s_i\}, \emptyset), (L_1, L_1'), (L_2, L_2') \ldots$$

  where $P_j = \{s \in S \mid p_0 \in I, p_0 \rightarrow^{\alpha(0,j)} s\}$, and L_j, L_j' are determinied by the definition of $\mathcal{A}'$.
- We show that r' is accepting. Assume otherwise, and let m be an index such that $L_n \neq L_n'$ for all $n \geq m$.
- Then let $j > m$ be some index with $s_j \in F$; hence $s_j \in L_j'$. There exists a $k > j$ such that $L_{k+1}' = \{s \in S \mid s_j \rightarrow^{\alpha(j,k)} s\} = \{s \in S \mid s_i \rightarrow^{\alpha(i,k)} s\} = L_{k+1}$.
- Contradiction.

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Lemma 2 *For every semi-deterministic Büchi automaton $\mathcal{A}$ there exists a deterministic Muller automaton $\mathcal{A}'$ with $\mathcal{L}(\mathcal{A}) = \mathcal{L}(\mathcal{A}')$.*

Proof:

Let $\mathcal{A} = (N \uplus D, I, T, F)$, $d = |D|$, and let D be ordered by $<$. We construct the DMA $(S', \{s_0'\}, T', \mathcal{F})$:

- $S' = 2^N \times \{0, \ldots, 2d\} \rightarrow D \cup \{\llcorner\!\lrcorner\}$

- $s'_0 = (\{N \cap I\}, (d_1, d_2, \ldots, d_n, \llcorner, \ldots, \llcorner))$,
  where $d_i < d_{i+1}, \{d_1, \ldots, d_n\} = D \cap I\}$.
- $T' = \{((N_1, f_1), \sigma, (N_2, f_2)) \mid N_2 = pr_3(T \cap N_1 \times \{\sigma\} \times N)$
  $D' = pr_3(T \cap N_1 \times \{\sigma\} \times D)$
  $g_1 : n \mapsto d_2 \in D \Leftrightarrow f_1 : n \mapsto d_1 \in D \wedge d_1 \to^{\sigma} d_2$
  g_2: insort the elements of D' in the empty slots of g_1 (using $<$)
  f_2: delete every recurrance (leaving an *empty* slot)
- $\mathcal{F} = \{F' \subseteq S' \mid \exists i \in 1, \ldots, 2d$ s.t.
  $f(i) \neq \llcorner$ for all $(N', f) \in F'$ and
  $f(i) \in F$ for some $(N', f) \in F'\}$.

(... to be continued.)

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