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Automata, Games and Verification: Lecture 4

Theorem 1 *For each Büchi automaton $\mathcal{A}$ there exists a Büchi automaton $\mathcal{A}'$ such that $\mathcal{L}(\mathcal{A}') = \Sigma^\omega \setminus \mathcal{L}(\mathcal{A})$.*

Helpful definitions:

- A *level ranking* is a function $g : S \to \{0, \ldots, 2 \cdot |S|\} \cup \{\bot\}$ such that if $g(s)$ is odd, then $s \notin F$.
- Let $\mathcal{R}$ be the set of all level rankings.
- A level ranking g' *covers* a level ranking g if, for all $s, s' \in S$, if $g(s) \geq 0$ and $(s, \sigma, s') \in T$, then $0 \leq g'(s') \leq g(s)$.

Proof:

We define $\mathcal{A}' = (S', I', T', F')$ with

- $S' = \mathcal{R} \times 2^S$;
- $I' = \{\langle g_0, \emptyset\rangle$, where $g_0(s) = 2 \cdot |S|$ if $s \in I$ and $g_0(s) = \bot$ if $s \notin I$;
- $T = \{(\langle g, \emptyset\rangle, \sigma, \langle g', P'\rangle) \mid g'$ covers g, and $P' = \{s' \in S \mid g'(s')$ is even $\}$
  $\cup\ \{(\langle g, P\rangle, \sigma, \langle g', P'\rangle) \mid P \neq \emptyset, g'$ covers g, and
  $P' = \{s' \in S \mid (s, \sigma, s') \in T, s \in P, g'(s')$ is even $\}$;
- $F = \mathcal{R} \times \{\emptyset\}$.

(Intuition: $\mathcal{A}'$ guesses the level rankings for the run DAG. The P component tracks the states whose corresponding vertices in the run DAG have even ranks. Paths that traverse such vertices should eventually reach a vertex with odd rank. The acceptance condition ensures that all paths visit a vertex with odd rank infinitely often.)

$\mathcal{L}(\mathcal{A}') \subseteq \Sigma^\omega \setminus \mathcal{L}(\mathcal{A})$:

- Let $\alpha \in \mathcal{L}(\mathcal{A}')$ and let $r' = (g_0, P_0), (g_1, P_1), \ldots$ be an accepting run of $\mathcal{A}'$ on α.
- Let $G = (V, E)$ be the run DAG of $\mathcal{A}$ on α.
- The function $f : \langle s, l\rangle \mapsto g_l(s), s \in S_l, l \in \omega$ is a ranking for G:
  - if $g_i(s)$ is odd then $s \notin F$;
  - for all $(\langle s, l\rangle, \langle s', l + 1\rangle) \in E$, $g_{l+1}(s') \leq g_l(s)$.
- f is an odd ranking:

  - Assume otherwise. Then there exists a path $\langle s_0, l_0\rangle, \langle s_1, l_1\rangle, \langle s_2, l_2\rangle, \ldots$ in G such that for infinitely many $i \in \omega$, $f(\langle s_i, l_i\rangle)$ is even.
  - Hence, there exists an index $j \in \omega$, such that $f(\langle s_j, l_j\rangle)$ is even and, for all $k \geq 0$, $f(\langle s_{j+k}, l_{j+k}\rangle) = f(\langle s_j, l_j\rangle)$.
  - Since r' is accepting, $P_{j'} = \emptyset$ for infinitely many j'. Let j' be the smallest such index $\geq j$.
  - $P_{j'+1+k} \neq \emptyset$ for all $k \geq 0$.
  - Contradiction.
- Since there exists an odd ranking, $\alpha \notin \mathcal{L}(\mathcal{A})$.

$\Sigma^\omega \setminus \mathcal{L}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{A}')$:

- Let $\alpha \in \Sigma^\omega \setminus \mathcal{L}(\mathcal{A})$ and let $G = (V, E)$ be the run DAG of $\mathcal{A}$ on α.
- There exists an odd ranking f on G.
- There is a run $r' = (g_0, P_0), (g_1, P_1), \ldots$ of $\mathcal{A}'$ on α, where

$$g_l(s) = \begin{cases} f(\langle s, l\rangle) & \text{if } s \in S_l; \\ \bot & \text{otherwise;} \end{cases}$$

$P_0 = \emptyset$,

$$P_{l+1} = \begin{cases} \{s \in S \mid g_{l+1}(s) \text{ is even } \} & \text{if } P_l = \emptyset, \\ \{s' \in S \mid \exists s \in S_l \cap P_l \,.\, (\langle s, l\rangle, \langle s', l+1\rangle) \in E, g_{l+1}(s') \text{ is even}\} & \text{otherwise.} \end{cases}$$

- r' is accepting. (Assume there is an index i such that $P_j \neq \emptyset$ for all $j \geq i$. Then there exists a path in G that visits an even rank infinitely often.)
- Hence, $\alpha \in \mathcal{L}(\mathcal{A}')$.

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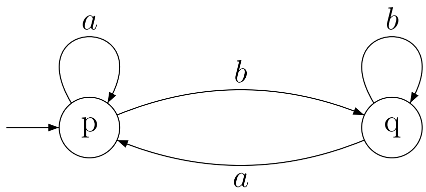
6 Muller Automata

Definition 1 *A* (nondeterministic) Muller automaton $\mathcal{A}$ *over alphabet* Σ *is a tuple* (S, I, T, F)*:*

- S, I, T *: defined as before*
- $\mathcal{F} \subseteq 2^S$ *: set of* accepting subsets*, called the* table*.*

Definition 2 *A run* r *of a Muller automaton is* accepting *iff* $In(r) \in F$

Example:



- for $\mathcal{F} = \{\{q\}\}$: $\mathcal{L}(\mathcal{A}) = (a \cup b)^* b^\omega$
- for $\mathcal{F} = \{\{q\}, \{p, q\}\}$: $\mathcal{L}(\mathcal{A}) = (a^* b)^\omega$

✚

Theorem 2 *For every (deterministic) Büchi automaton $\mathcal{A}$, there is (deterministic) Muller automaton $\mathcal{A}'$, such that $\mathcal{L}(\mathcal{A}) = \mathcal{L}(\mathcal{A}')$.*

Proof:

$$S' = S,\ I' = I,\ T' = T$$
$$\mathcal{F}' = \{Q \subseteq S \mid Q \cap F \neq \emptyset\}$$

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Theorem 3 *For every nondeterministic Muller automaton $\mathcal{A}$ there is a nondeterministic Büchi automaton $\mathcal{A}'$ such that $\mathcal{L}(\mathcal{A}) = \mathcal{L}(\mathcal{A}')$.*

Proof:

- $\mathcal{F} = \{F_1, \ldots, F_n\}$
- $S' = S \cup \bigcup_{i=1}^{n} \{i\} \times F_i \times 2^{F_i}$
- $I' = I$
- $T' = T$
  $\cup\ \{(s, \sigma, (i, s', \emptyset)) | 1 \leq i \leq n, (s, \sigma, s') \in T, s' \in F_i\}$
  $\cup\ \{((i, s, R), \sigma, (i', s', R')) \mid 1 \leq i \leq n, s, s' \in F_i, R, R' \subseteq F_i,$
  $(s, \sigma, s') \in T, R' = R \cup \{s\}$ if $R \neq F_i$ and $R' = \emptyset$ if $R = F_i\}$
- $F' = \bigcup_{i=1}^{n} \{i\} \times F_i \times \{F_i\}$

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Boolean language operations: complementation, union, intersection.

Theorem 4 *The languages recognizable by deterministic Muller automata are closed under boolean operations.*

Proof:

- $\mathcal{L}(\mathcal{A}') = \Sigma^\omega \setminus \mathcal{L}(\mathcal{A})$:
  - $S' = S, I' = I, T' = T, \mathcal{F}' = 2^S \setminus \mathcal{F}$
- $\mathcal{L}(\mathcal{A}') = \mathcal{L}(\mathcal{A}_1) \cap \mathcal{L}(\mathcal{A}_2)$:
  - $S' = S_1 \times S_2, I' = I_1 \times I_2$,
  - $T' = \{((s_1, s_2), \sigma, (s_1', s_2')) \mid (s_1, \sigma, s_1') \in T_1, (s_2, \sigma, s_2') \in T_2\}$
  - $\mathcal{F}' = \{\{(p_1, q_1), \ldots, (p_n, q_n)\} \mid \{p_1, \ldots, p_n\} \in \mathcal{F}_1, \{q_1, \ldots, q_n\} \in \mathcal{F}_2\}$
- $\mathcal{L}(\mathcal{A}_1) \cup \mathcal{L}(\mathcal{A}_2) = \Sigma^\omega \setminus ((\Sigma^\omega \setminus \mathcal{L}(\mathcal{A}_1)) \cap (\Sigma^\omega \setminus \mathcal{L}(\mathcal{A}_2)))$.

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Theorem 5 *A language $\mathcal{L}$ is recognizable by a deterministic Muller automaton iff $\mathcal{L}$ is a boolean combination of languages $\overrightarrow{W}$ where $W \subseteq \Sigma^*$ is regular.*

Proof:

($\Leftarrow$)

- If W is regular, then $\overrightarrow{W}$ is recognizable by a deterministic Büchi automaton;
- hence, $\overrightarrow{W}$ is recognizable by a deterministic Muller automaton;
- hence, the boolean combination $\mathcal{L}$ is recognizable by a deterministic Muller automaton.

($\Rightarrow$) *left as an exercise.*

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