Bernd Finkbeiner
Sven Schewe

Date: May 8, 2008

Automata, Games and Verification: Lecture 3

Definition 1 (Substrings) *Let $\alpha \in \Sigma^*$. For two integers $n \leq m$ we define*

$$\alpha(n, m) = \alpha(n)\alpha(n+1)\ldots\alpha(m) \ .$$

Definition 2 (Limit) *For $W \subseteq \Sigma^*$:*

$$\overrightarrow{W} = \{\alpha \in \Sigma^\omega \mid \text{there exist infinitely many } n \in \omega \text{ s.t. } \alpha(0, n) \in W\} \ .$$

Theorem 1 *An ω-language $L \subseteq \Sigma^\omega$ is recognizable by a deterministic Büchi automaton iff there is a regular language $W \subseteq \Sigma^*$ s.t. $L = \overrightarrow{W}$.*

Proof:

Let L be the language of a deterministic Büchi automaton $\mathcal{A}$; let W be the regular language of $\mathcal{A}$ as a deterministic finite-word automaton. We show that $L = \overrightarrow{W}$.

$$\begin{array}{ll} & \alpha \in L \\ \text{iff} & \text{for the unique run } r \text{ of } \mathcal{A} \text{ on } \alpha,\ In(r) \cap F \neq \emptyset \\ \text{iff} & \alpha(0, n) \in W \text{ for infinitely many } n \in \omega \\ \text{iff} & \alpha \in \overrightarrow{W}. \end{array}$$

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5 Complementation

Theorem 2 *For any deterministic Büchi automaton $\mathcal{A}$, there exists a Büchi automaton $\mathcal{A}'$ such that $\mathcal{L}(\mathcal{A}') = \Sigma^\omega \smallsetminus \mathcal{L}(\mathcal{A})$.*

Proof:

We construct $\mathcal{A}'$ as follows:

- $S' = (S \times \{0\}) \cup ((S \smallsetminus F) \times \{1\})$.
- $I' = I \times \{0\}$.
- $T' = \{((s, 0), \sigma, (s', 0)) \mid (s, \sigma, s') \in T\} \cup \{((s, 0), \sigma, (s', 1)) \mid (s, \sigma, s') \in T\} \cup \{((s, 1), \sigma, (s, 1)) \mid (s, \sigma, s') \in T, s' \in S - F\}$.
- $F' = (S - F) \times \{1\}$.

$\mathcal{L}(\mathcal{A}') \subseteq \Sigma^\omega - \mathcal{L}(\mathcal{A})$:

- For $\alpha \in \mathcal{L}(\mathcal{A}')$ we have an accepting run

$$r' : (s_0, 0)(s_1, 0) \dots (s_j, 0)(s'_0, 1)(s'_1, 1) \dots$$

  on $\mathcal{A}'$.

- Hence,

$$r : s_0 s_1 s_2 \dots s_j s'_0 s'_1 \dots$$

  is the unique run on α in $\mathcal{A}$.

- Since $s'_0, s'_1, \ldots \in S \setminus F$, $In(r) \subseteq S \setminus F$. Hence, r is not accepting and $\alpha \in \Sigma^\omega - \mathcal{L}(\mathcal{A})$

$\mathcal{L}(\mathcal{A}') \supseteq \Sigma^\omega - \mathcal{L}(\mathcal{A})$:

- We assume $\alpha \notin \mathcal{L}(\mathcal{A})$. Since $\mathcal{A}$ is deterministic and complete there exists a run

$$r : s_0 s_1 s_2 \dots$$

  for α on $\mathcal{A}$, but $In(r) \cap F = \emptyset$.

- Thus there exists a $k \in \omega$ such that $s_j \notin F$ for $j > k$.
- This gives us the run

$$r' : (s_0, 0)(s_1, 0) \dots (s_k, 0)(s_{k+1}, 1)(s_{k+2}, 1) \dots$$

  for α on $\mathcal{A}'$ with the property $In(r') \subseteq ((S - F) \times \{1\}) = F'$.

- Hence, r' is accepting and $\alpha \in \mathcal{L}(\mathcal{A}')$.

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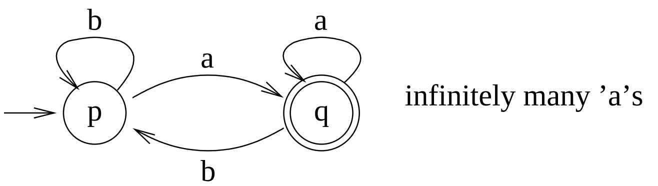
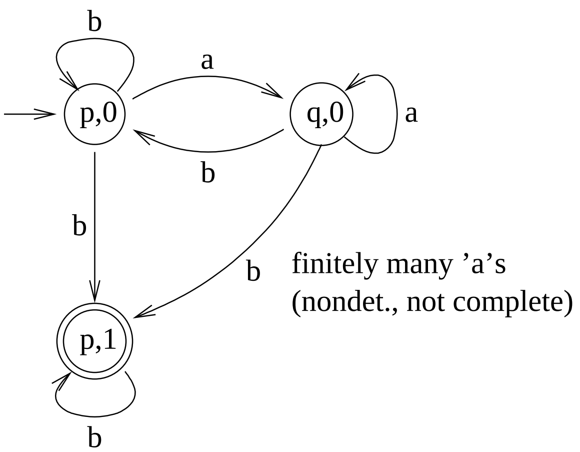
Example:





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Reference: The following construction for the complementation of nondeterministic Büchi automata is taken from: Orna Kupferman and Moshe Y. Vardi, Weak alternating automata are not that weak. *ACM Trans. Comput. Logic* 2, 3 (Jul. 2001), 408-429.

Definition 3 *Let* $\mathcal{A} = (S, I, T, F)$ *be a nondeterministic Büchi automaton. The* run DAG *of* $\mathcal{A}$ *on a word* $\alpha \in \Sigma^\omega$ *is the directed acyclic graph* $G = (V, E)$ *where*

- $V = \bigcup_{l \geq 0}(S_l \times \{l\})$ *where* $S_0 = I$ *and* $s_{l+1} = \bigcup_{s \in S_l, (s, \alpha(l), s') \in T}\{s'\}$
- $E = \{(\langle s, l\rangle, \langle s', l+1\rangle) \mid l \geq 0, (s, \alpha(l), s') \in T\}$

A path in a run DAG is accepting iff it visits F infinitely often. The automaton accepts α if some path is accepting.

Definition 4 *A* ranking *for* G *is a function* $f : V \to \{0, \ldots, 2 \cdot |S|\}$ *such that*

- *for all* $\langle s, l\rangle \in V$*, if* $f(\langle s, l\rangle)$ *is odd then* $s \notin F$*;*
- *for all* $(\langle s, l\rangle, \langle s', l'\rangle) \in E$*,* $f(\langle s', l'\rangle) \leq f(\langle s, l\rangle)$*.*

A ranking is *odd* iff for all paths $\langle s_0, l_0\rangle, \langle s_1, l_1\rangle, \langle s_2, l_2\rangle, \ldots$ in G, there is a $i \geq 0$ such that $f(\langle s_i, l_i\rangle)$ is odd and, for all $j \geq 0$, $f(\langle s_{i+j}, l_{i+j}\rangle) = f(\langle s_i, l_i\rangle)$.

Lemma 1 *If there exists an odd ranking for* G*, then* $\mathcal{A}$ *does not accept* α*.*

Proof:

- In an odd ranking, every path eventually gets trapped in a some odd rank.
- If $f(\langle s, l\rangle)$ is odd, then $s \notin F$.
- Hence, every path visits F only finitely often.

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Let G' be a subgraph of G. We call a vertex $\langle s, l\rangle$

- *safe* in G' if for all vertices $\langle s', l'\rangle$ reachable from $\langle s, l\rangle$, $s' \notin F$, and
- *endangered* in G' if only finitely many vertices are reachable.

We define an infinite sequence $G_0 \supseteq G_1 \supseteq G_2 \supseteq \ldots$ of DAGs inductively as follows:

- $G_0 = G$
- $G_{2i+1} = G_{2i} \setminus \{\langle s, l\rangle \mid \langle s, l\rangle \text{ is endangered in } G_{2i}\}$
- $G_{2i+2} = G_{2i+1} \setminus \{\langle s, l\rangle \mid \langle s, l\rangle \text{ is safe in } G_{2i}\}$.

Lemma 2 *If* $\mathcal{A}$ *does not accept* α*, then the following holds: For every* $i \geq 0$ *there exists an* l_i *such that for all* $j \geq l_i$ *at most* $|S| - i$ *vertices of the form* $\langle \_, j\rangle$ *are in* G_{2i}*.*

Proof:

Proof by induction on i:

- $i = 0$: In G, for every l, there are at most $|S|$ vertices of the form $\langle \_, l\rangle$.
- $i \to i+1$:
  - Case G_{2i} is finite: then $G_{2(i+1)}$ is empty.
  - Case G_{2i} is infinite:
    * There must exist a safe vertex $\langle s, l\rangle$ in G_{2i+1}. (Otherwise, we can construct a path in G with infinitely many visits to F).
    * We choose $l_{i+1} = l$.
    * We prove that for all $j \geq l$, there are at most $|S| - (i+1)$ vertices of the form $\langle \_, j\rangle$ in G_{2i+2}.
      · Since $\langle s, l\rangle \in G_{2i+1}$, it is not endangered in G_{2i}.
      · Hence, there are infinitely many vertices reachable from $\langle s, l\rangle$ in G_{2i}.
      · By König's Lemma, there exists an infinite path $p = \langle s, l\rangle, \langle s_1, l+1\rangle, \langle s, l+2\rangle, \ldots$ in G_{2i}.
      · No vertex on p is endangered (there is an infinite path). Therefore, p is in G_{2i+1}.
      · All vertices on p are safe ($\langle s, l\rangle$ is safe) in G_{2i+1}. Therefore, none of the vertices on p are in G_{2i+2}.
      · Hence, for all $j \geq l$, the number of vertices of the form $\langle \_, l\rangle$ is strictly smaller than their number in G_{2i}.

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Lemma 3 *If $\mathcal{A}$ does not accept α, then there exists an odd ranking for G.*

Proof:

- We define $f(\langle s, l\rangle) = 2i$ if $\langle s, l\rangle$ is endangered in G_{2i} and
- $f(\langle s, l\rangle) = 2i+1$ if $\langle s, l\rangle$ is safe in G_{2i}.
- f is a ranking:
  - by Lemma 2, G_j is empty for $j > 2 \cdot |S|$. Hence, $f : V \to \{0, \ldots, 2 \cdot |S|\}$.
  - if $\langle s', l'\rangle$ is a successor of $\langle s, l\rangle$, then $f(\langle s', l'\rangle) \leq f(\langle s, l\rangle)$
    * Let $j := f(\langle s, l\rangle)$.
    * Case j is even: vertex $\langle s, l\rangle$ is endangered in G_j; hence either $\langle s', l'\rangle$ is not in G_j, and therefore $f(\langle s, l\rangle) < j$; or $\langle s', l'\rangle$ is in G_j and endangered; hence, $f(\langle s, l\rangle) = j$.
    * Case j is odd: vertex $\langle s, l\rangle$ is safe in G_j; hence either $\langle s', l'\rangle$ is not in G_j, and therefore $f(\langle s, l\rangle) < j$; or $\langle s', l'\rangle$ is in G_j and safe; hence, $f(\langle s, l\rangle) = j$.
  - f is an odd ranking:
    * For every path $\langle s_0, l_0\rangle, \langle s_1, l_1\rangle, \langle s_2, l_2\rangle, \ldots$ in G there exists an $i \geq 0$ such that for all $j \geq 0$, $f(\langle s_{i+j}, l_{i+j}\rangle) = f(\langle s_i, l_i\rangle)$.

* Suppose that $k := f(\langle s_i, l_i \rangle)$ is even. Thus, $\langle s_i, l_i \rangle$ is endangered in G_k.
* Since $f(\langle s_{i+j}, l_{i+j} \rangle) = k$ for all $j \geq 0$, all $\langle s_{i+j}, l_{i+j} \rangle$ are in G_k.
* This contradicts that $\langle s_i, l_i \rangle$ is endangered in G_k.

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