Down the Borel Hierarchy: Solving Muller Games via Safety Games*

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***Abstract*—We transform a Muller game with n vertices into a safety game with $(n!)^3$ vertices whose solution allows to determine the winning regions of the Muller game and to compute a finite-state winning strategy for one player. This yields a novel memory structure and a natural notion of permissive strategies for Muller games. Moreover, we generalize our construction by presenting a new type of game reduction from infinite games to safety games and show its applicability to several other winning conditions.**

I. Introduction

Muller games are a source of interesting and challenging questions in the theory of infinite games. They are expressive enough to describe all ω-regular properties. Also, all winning conditions that depend only on the set of vertices visited infinitely often can trivially be reduced to Muller games. Hence, they subsume Büchi, co-Büchi, parity, Rabin, and Streett conditions. Furthermore, Muller games are not positionally determined, i.e., both players need memory to implement their winning strategies. In this work, we present a framework to deal with three aspects of Muller games: solution algorithms, memory structures, and quality measures for strategies.

While investigating the interest of Muller games for "casual living-room recreation" [1], McNaughton introduced scoring functions which describe the progress a player is making towards winning a play: consider a Muller game $(\mathcal{A}, \mathcal{F}_0, \mathcal{F}_1)$, where $\mathcal{A}$ is the arena and $(\mathcal{F}_0, \mathcal{F}_1)$ is a partition of the set of loops in $\mathcal{A}$ used to determine the winner: Player i wins a play ρ if the set of vertices visited infinitely often by ρ is in $\mathcal{F}_i$. The score of a set F of vertices measures how often F has been visited completely since the last visit of a vertex not in F. McNaughton proved the existence of strategies for the winning player that bound her opponent's scores by $|\mathcal{A}|!$ [1], provided the play starts in her winning region. Such a strategy is necessarily winning. The bound $|\mathcal{A}|!$ was subsequently improved to 2 (and shown to be tight) [2]. Thus, the winner of a Muller game can be determined by solving a (much simpler, albeit large) safety game. In the following, we present a novel algorithm and a novel type of memory structure for Muller games derived from solving this safety game. We also obtain a natural quality measure for strategies in Muller games and are able to extend the definition of permissiveness [3] from parity games to Muller games.

In the following, we use the notions of winning strategies and winning regions as defined in [4].

II. Scoring Functions for Muller Games

We begin by introducing scoring functions. For a more detailed treatment we refer to [2], [1].

Definition 1. *Let $w \in V^*$, $v \in V$, and $\emptyset \neq F \subseteq V$.*

- *Define* $\mathrm{Sc}_F(\varepsilon) = 0$.
- *If $v \notin F$, then $\mathrm{Sc}_F(wv) = 0$ and $\mathrm{Acc}_F(wv) = \emptyset$.*
- *If $v \in F$ and $\mathrm{Acc}_F(w) = F \setminus \{v\}$, then $\mathrm{Sc}_F(wv) = \mathrm{Sc}_F(w) + 1$ and $\mathrm{Acc}_F(wv) = \emptyset$.*
- *If $v \in F$ and $\mathrm{Acc}_F(w) \neq F \setminus \{v\}$, then $\mathrm{Sc}_F(wv) = \mathrm{Sc}_F(w)$ and $\mathrm{Acc}_F(wv) = \mathrm{Acc}_F(w) \cup \{v\}$.*

Now, let $w, w' \in V^*$ and $\mathcal{F} \subseteq 2^V$.

1) *w is $\mathcal{F}$-smaller than w', denoted by $w \leq_{\mathcal{F}} w'$, if $\mathrm{Last}(w) = \mathrm{Last}(w')$ and for all $F \in \mathcal{F}$:*
   - *$\mathrm{Sc}_F(w) < \mathrm{Sc}_F(w')$, or*
   - *$\mathrm{Sc}_F(w) = \mathrm{Sc}_F(w')$ and $\mathrm{Acc}_F(w) \subseteq \mathrm{Acc}_F(w')$.*
2) *w and w' are $\mathcal{F}$-equivalent, denoted by $w =_{\mathcal{F}} w'$, if $w \leq_{\mathcal{F}} w'$ and $w' \leq_{\mathcal{F}} w$.*

Our results rely on the following lemma.

Lemma 1 ([2]). *In every Muller game $\mathcal{G} = (\mathcal{A}, \mathcal{F}_0, \mathcal{F}_1)$, Player i has a winning strategy that bounds every Sc_F with $F \in \mathcal{F}_{1-i}$ by two during every consistent play.*

Hence, a player wins the Muller game if and only if she can prevent her opponent from ever reaching a score of three. This is a safety condition!

III. Solving Muller by Solving Safety

Fix a Muller game $\mathcal{G} = (\mathcal{A}, \mathcal{F}_0, \mathcal{F}_1)$ and consider the following safety game $\mathcal{G}_S$: the scores and accumulators of Player 1 are tracked up to threshold three by the arena. More formally, we take the $=_{\mathcal{F}_1}$-quotient of the unraveling of $\mathcal{A}$ up to the positions where Player 1 reaches a score of three for the first time. Player 1 wins a play in this (finite) arena, if he reaches a score of three. Hence, Player 0 wins if her opponent never reaches a score of three.

*This work was supported by the projects *Games for Analysis and Synthesis of Interactive Computational Systems (GASICS)* and *Logic for Interaction (LINT)* of the *European Science Foundation*.

Theorem 1. *Let $\mathcal{G}$ be a Muller game with vertex set V. One can effectively construct a safety game $\mathcal{G}_S$ with vertex set V^S and a mapping $f\colon V \to V^S$ with the following properties:*

1) *For every $v \in V$: Player i wins the Muller game from v if and only if she wins the safety game from $f(v)$.*
2) *Player 0 has a finite-state winning strategy for $\mathcal{G}$ whose set of memory states is V^S.*
3) $|V^S| \leq (|V|!)^3$.

Note that the first statement speaks about both players while the second one only speaks about Player 0. This is due to the fact that the safety game keeps track of Player 1's scores only. To obtain a winning strategy for Player 1, we have to track Player 0's scores. The first claim follows directly from Lemma 1 while the second one is proved by turning the winning region of Player 0 in $\mathcal{G}_S$ (restricted to the vertices reachable via a positional winning strategy for $\mathcal{G}_S$) into a memory structure whose strategy prevents Player 1 from reaching a score of three in $\mathcal{G}$. Such a strategy is winning. The size of this memory structure is at most cubically larger then the size of the LAR memory structure.

Furthermore, by only using the $\leq_{\mathcal{F}_1}$-maximal elements of Player 0's winning region as memory states, one obtains an even smaller memory structure that still implements a winning strategy. On the other hand, by using all vertices in the winning region, but using the most general non-deterministic winning strategy for Player 0 in $\mathcal{G}_S$ (cf. [3]), we also obtain the most general non-deterministic winning strategy that prevents the losing player from reaching a score of three (which can obviously be generalized to any threshold k). This extends the notion of permissive strategies from parity to Muller games.

IV. Safety Reductions for Infinite Games

Since Muller conditions are on a higher level of the Borel hierarchy than safety conditions, there is no game reduction from Muller to safety games (using the notion of reduction as defined, e.g., in [4]). Nonetheless, we have just solved a Muller game by solving a safety game. The price we have to pay is that we only obtain a winning strategy for one player while *standard* reductions yield winning strategies for both. Next, we present a general construction comprising our result.

Definition 2. *A game $\mathcal{G} = (\mathcal{A}, \mathrm{Win})$ with vertex set V and set $\mathrm{Win} \subseteq V^\omega$ of winning plays for Player 0 is (finite-state) safety reducible, if there is a regular language $L \subseteq V^*$ of finite words such that:*

- *For every play $\rho \in V^\omega$: if $\mathrm{Pref}(\rho) \subseteq L$, then $\rho \in \mathrm{Win}$.*
- *If Player 0 wins from v, then she has a strategy σ such that $\mathrm{Pref}(\rho) \subseteq L$ for every ρ consistent with σ and starting in v.*

Note that a strategy σ satisfying the second property is winning for Player 0 from v. Many solution algorithms for games can be phrased in this terminology, e.g., the progress measure algorithms for parity games [5] respectively Rabin and Streett games [6], as well as work on bounded synthesis [7] and LTL realizability [8].

Theorem 2. *Let $\mathcal{G}$ be a game with vertex set V that is safety reducible with language $L(\mathfrak{A})$ for some DFA $\mathfrak{A} = (Q, V, q_0, \delta, F)$. Define the safety game $\mathcal{G}' = (\mathcal{A} \times \mathfrak{A}, V \times F)$.*

1) *For every $v \in V$, Player 0 wins the $\mathcal{G}$ from v if and only if she wins $\mathcal{G}'$ from $(v, \delta(q_0, v))$.*
2) *Player 0 has a finite-state winning strategy for $\mathcal{G}$ with memory states Q.*

This results gives a unified approach to solving parity, Rabin, Streett, and Muller games (and many more) by solving safety games. Furthermore, the notion of safety-reduction allows to generalize permissiveness to all these games, yielding what one could call L-permissiveness, i.e., we obtain the most general non-deterministic winning strategy that "stays" in L.

V. Conclusion

We have shown how to translate a Muller game into a safety game to determine both winning regions and a finite-state winning strategy for one player. Then, we generalized this construction to a new type of reduction from infinite games to safety games with the same properties. The reduction from Muller to safety games is implemented in the tool `GAVS+`[1] [9].

The quality of a strategy can be measured by the maximal score value the opponent can achieve. We conjecture that there is no tradeoff between size and quality of a strategy.

Finally, there is a tight connection between permissive strategies, progress measure algorithms, and safety reductions for parity games. Whether the safety reducibility of Muller games can be turned into a progress measure algorithm is subject to ongoing research.

Acknowledgments. The authors want to thank Wolfgang Thomas for bringing McNaughton's work to their attention, Wladimir Fridman for fruitful discussions, and Chih-Hong Cheng for his implementation of the algorithm.

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[1] See www6.in.tum.de/~chengch/gavs/ for details and to download the tool.