Down the Borel Hierarchy: Solving Muller Games via Safety Games*

Daniel Neider*, Roman Rabinovich†, and Martin Zimmermann*†

* Lehrstuhl für Informatik 7, RWTH Aachen University, Germany
Email: {neider, zimmermann}@automata.rwth-aachen.de
† Mathematische Grundlagen der Informatik, RWTH Aachen University, Germany
Email: rabinovich@logic.rwth-aachen.de
‡ Institute of Informatics, University of Warsaw, Poland

Abstract—We transform a Muller game with \( n \) vertices into a safety game with \((n!)^3\) vertices whose solution allows to determine the winning regions of the Muller game and to compute a finite-state winning strategy for one player. This yields a novel memory structure and a natural notion of permissive strategies for Muller games. Moreover, we generalize our construction by presenting a new type of game reduction from infinite games to finite-state winning strategy for one player. This yields a novel mine the winning regions of the Muller game and to compute a natural quality measure for strategies in Muller games and are able to extend the definition of permissiveness [3] from parity games to Muller games.

In the following, we use the notions of winning strategies and winning regions as defined in [4].

II. SCORING FUNCTIONS FOR MULLER GAMES

We begin by introducing scoring functions. For a more detailed treatment we refer to [2], [1].

Definition 1. Let \( w \in V^* \), \( v \in V \), and \( \emptyset \neq F \subseteq V \).

\[
\begin{align*}
\bullet & \quad \text{Define } \text{Sc}_F(w) = 0, \\
\bullet & \quad \text{If } v \notin F, \text{ then } \text{Sc}_F(wv) = 0 \text{ and } \text{Acc}_F(wv) = \emptyset, \\
\bullet & \quad \text{If } v \in F \text{ and } \text{Acc}_F(w) = F \setminus \{v\}, \text{ then } \text{Sc}_F(wv) = \text{Sc}_F(w) + 1 \text{ and } \text{Acc}_F(wv) = \emptyset, \\
\bullet & \quad \text{If } v \in F \text{ and } \text{Acc}_F(w) \neq F \setminus \{v\}, \text{ then } \text{Sc}_F(wv) = \text{Sc}_F(w) \text{ and } \text{Acc}_F(wv) = \text{Acc}_F(w) \cup \{v\}.
\end{align*}
\]

Now, let \( w, w' \in V^* \) and \( F \subseteq 2^V \).

1) \( w \) is \( F \)-smaller than \( w' \), denoted by \( w \leq_F w' \), if \( \text{Last}(w) = \text{Last}(w') \) and for all \( F' \in F \):

\[
\begin{align*}
\bullet & \quad \text{Sc}_F(w) < \text{Sc}_F(w'), \text{ or } \\
\bullet & \quad \text{Sc}_F(w) = \text{Sc}_F(w') \text{ and } \text{Acc}_F(w) \subseteq \text{Acc}_F(w').
\end{align*}
\]

2) \( w \) and \( w' \) are \( F \)-equivalent, denoted by \( w =_F w' \), if \( w \leq_F w' \) and \( w' \leq_F w \).

Our results rely on the following lemma.

Lemma 1 (2). In every Muller game \( G = (A, F_0, F_1) \). Player \( i \) has a winning strategy that bounds every \( \text{Sc}_F \) with \( F \in F_{1-i} \) by two during every consistent play.

Hence, a player wins the Muller game if and only if she can prevent her opponent from ever reaching a score of three. This is a safety condition!

III. SOLVING MULLER BY SOLVING SAFETY

Fix a Muller game \( G = (A, F_0, F_1) \) and consider the following safety game \( G_s \): the scores and accumulators of Player 1 are tracked up to threshold three by the arena. More formally, we take the \( =_{F_1} \)-quotient of the unraveling of \( A \) up to the positions where Player 1 reaches a score of three for the first time. Player 1 wins a play in this (finite) arena, if he reaches a score of three. Hence, Player 0 wins if her opponent never reaches a score of three.
Let $G$ be a Muller game with vertex set $V$. One can effectively construct a safety game $G_S$ with vertex set $V_S$ and a mapping $f : V \rightarrow V_S$ with the following properties:

1. For every $v \in V$: Player 1 wins the Muller game from $v$ if and only if she wins the safety game from $f(v)$.
2. Player 0 has a finite-state winning strategy for $G_S$ whose set of memory states is $V_S$.
3. $|V_S| \leq (|V|!)^3$.

Note that the first statement speaks about both players while the second one only speaks about Player 0. This is due to the fact that the safety game keeps track of Player 1’s scores only. To obtain a winning strategy for Player 1, we have to track Player 0’s scores. The first claim follows directly from Lemma 1 while the second one is proved by turning the winning region of Player 0 in $G_S$ (restricted to the vertices reachable via a positional winning strategy for $G_S$) into a memory structure whose strategy prevents Player 1 from reaching a score of three in $G$. Such a strategy is winning. The size of this memory structure is at most cubically larger then the size of the LAR memory structure.

Furthermore, by only using the $\leq_{xi}$-maximal elements of Player 0’s winning region as memory states, one obtains an even smaller memory structure that still implements a winning strategy. On the other hand, by using all vertices in the winning region, but using the most general non-deterministic winning strategy for Player 0 in $G_S$ (cf. [3]), we also obtain the most general non-deterministic winning strategy that prevents the losing player from reaching a score of three (which can obviously be generalized to any threshold $k$). This extends the notion of permissive strategies from parity to Muller games.

IV. SAFETY REDUCTIONS FOR INFINITE GAMES

Since Muller conditions are on a higher level of the Borel hierarchy than safety conditions, there is no game reduction from Muller to safety games (using the notion of reduction as defined, e.g., in [4]). Nonetheless, we have just solved a Muller game by solving a safety game. The price we have to pay is that we only obtain a winning strategy for one player while standard reductions yield winning strategies for both. Next, we present a general construction comprising our result.

**Definition 2.** A game $G = (A, \text{Win})$ with vertex set $V$ and set $\text{Win} \subseteq V^\omega$ of winning plays for Player 0 is (finite-state) safety reducible, if there is a regular language $L \subseteq V^*$ of finite words such that:

- For every play $\rho \in V^\omega$: if $\text{Pref}(\rho) \not\subseteq L$, then $\rho \in \text{Win}$.
- If Player 0 wins from $v$, then she has a strategy $\sigma$ such that $\text{Pref}(\rho) \subseteq L$ for every $\rho$ consistent with $\sigma$ and starting in $v$.

Note that a strategy $\sigma$ satisfying the second property is winning for Player 0 from $v$. Many solution algorithms for games can be phrased in this terminology, e.g., the progress measure algorithms for parity games [5] respectively Rabin and Streett games [6], as well as work on bounded synthesis [7] and LTL realizability [8].