The Complexity of Counting Models of Linear-time Temporal Logic

Joint work with Hazem Torfah

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Counting Complexity

- $f : \Sigma^* \to \mathbb{N}$ is in $\#\mathrm{P}$ if there is an NP machine $\mathcal{M}$ such that $f(w)$ is equal to the number of accepting runs of $\mathcal{M}$ on w.

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For complexity class $\mathcal{C}$:

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- Analogously: $\#_d\textsc{Exptime}$, $\#_d\textsc{Expspace}$, and $\#_d2\textsc{Exptime}$.

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Lemma

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Lemma

$\#_d$Pspace

#P ⊆ #Pspace ⊆ #Exptime ⊆ #NExptime ⊆ #Expspace ⊆ #2Exptime

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$\#_d\text{PSPACE} \subseteq \#_d\text{EXPTIME}$

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- f is $\#\mathrm{P}$-hard, if there is a polynomial time computable function r s. t. $f(r(\mathcal{M}, w))$ is equal to the number of accepting runs of $\mathcal{M}$ on w.

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- f is $\#\mathrm{P}$-hard, if there is a polynomial time computable function r s. t. $f(r(\mathcal{M}, w))$ is equal to the number of accepting runs of $\mathcal{M}$ on w.
- Hardness for other classes analogously.
- Completeness as usual.

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- *The following problem is $\#\mathrm{P}$-complete: Given an LTL formula φ and a bound k (in unary), how many k-word-models does φ have?*

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Upper bound: Guess word of length k and model-check it

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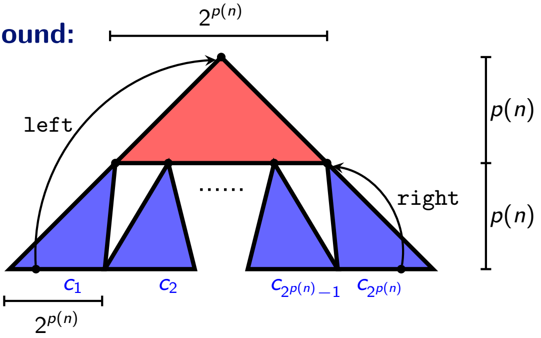

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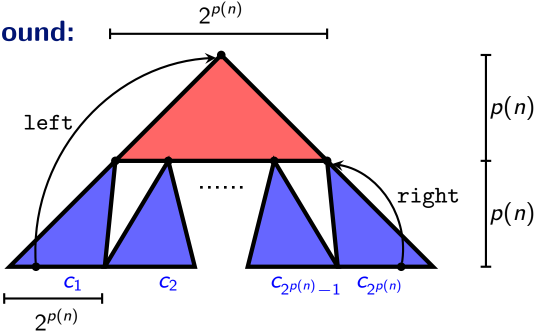

- **Upper bound:** Guess tree of height k and model-check it.

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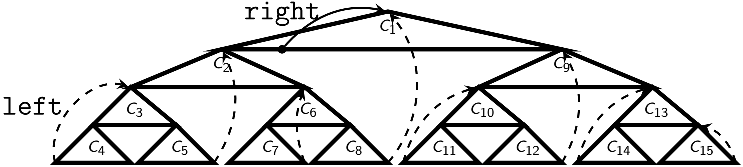

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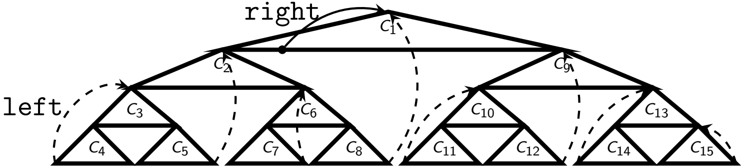

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  - Lowering the upper bound: how to guess and model-check doubly-exponentially sized trees in exponential space?
  - Raising the lower bound: how to encode doubly-exponentially sized configurations using polynomially sized formulas? Do games help?