Cost-Parity and Cost-Streett Games

Joint work with Nathanaël Fijalkow
(LIAFA & University of Warsaw)

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Algosyn Seminar, Aachen
Introduction

Boundedness problems in automata theory

- Star-height problem, finite power problem
- Automata with counters: BS-automata, max-automata, R-automata
- Logics with bounds: MSO+U, Cost-MSO
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- Finitary games: bounds between requests and responses
- Consumption and energy games: resources are consumed and recharged along edges
- Use automata with counters as winning conditions
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Here: an extension of $\omega$-regular and finitary games
Outline

1. Cost-Parity Games

2. Cost-Streett Games

3. Conclusion
Parity Games and Extensions

Games are played in arena $G$ colored by $\Omega: V \to \mathbb{N}$

Parity condition: Player 0 wins play $\iff$ maximal color seen infinitely often is even
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Parity condition: Player 0 wins play $\iff$ maximal color seen infinitely often is even

Equivalently:

- Request: vertex of odd color
- Response: vertex of larger even color
- Parity condition: almost all requests are answered
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- Player 0 wins since only finitely many requests are seen
- Player 1 wins since he can stay longer and longer in loop
From Cost-Parity to Bounded Cost-Parity

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**Lemma**

Let $C = (G, \text{CostParity}(\Omega))$ and let $B = (G, \text{BndCostParity}(\Omega))$.

1. $W_0(B) \subseteq W_0(C)$.
2. If $W_0(B) = \emptyset$, then $W_0(C) = \emptyset$.

**Corollary**

"To solve cost-parity games, it suffices to solve bounded cost-parity games."
From Bounded Cost-Parity to $\omega$-regular

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- Parity$(\Omega)$: plays satisfying the parity condition
- FinCost: plays with finite cost
- RR$(\Omega)$: plays in which every request is answered

$$PFRR(\Omega) = (\text{Parity}(\Omega) \cap \text{FinCost}) \cup \text{RR}(\Omega)$$
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**Lemma**

Let $B = (G, \text{BndCostParity}(\Omega))$, and let $P = (G, PFRR(\Omega))$. Then, $W_i(B) = W_i(P)$ for $i \in \{0, 1\}$. 
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- $\text{PFRR}(\Omega)$ is $\omega$-regular
- $\mathcal{P}$ can be reduced to parity game using small memory
- Thus, small finite-state winning strategies for both players in $\mathcal{P}$
Theorem
Given an algorithm that solves parity games in time $T(n, m, d)$, there is an algorithm that solves cost-parity games in time $O(n \cdot T(d \cdot n, d \cdot m, d + 2))$. 
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Theorem

The following problem is in $\text{NP} \cap \text{coNP}$: given a cost-parity game $G$ and a vertex $v$, has Player 0 a winning strategy from $v$?
Recall: Player 0 has finite state winning strategy $\sigma$ in (bounded) cost-parity game

**Theorem**

*Player 0 has positional winning strategies in (bounded) cost-parity games.*
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**Idea:** use quality measure $\text{Sh}: V^+ \to (D, \leq)$ for play prefixes with:

- $(D, \leq)$ is total order
- $\text{Sh}$ is congruence, i.e., $\text{Sh}(x) \leq \text{Sh}(y) \implies \text{Sh}(xv) \leq \text{Sh}(yv)$
- $\{\text{Sh}(w) \mid w \sqsubseteq \rho\}$ is finite $\implies \rho$ is winning or Player 0
- Finite-state strategies only allow plays $\rho$ s.t. $\{\text{Sh}(w) \mid w \sqsubseteq \rho\}$ is finite

**Half-positional Determinacy**

Martin Zimmermann  University of Warsaw  Cost-Parity and Cost-Streett Games  10/15
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Positional winning strategy: always play like you are in the worst situation possible that is consistent with $\sigma$
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Requests: sets of vertices $Q_i$ for $i = 1, \ldots, d$
Responses: sets of vertices $P_i$ for $i = 1, \ldots, d$
Cost functions for every pair $(Q_i, P_i)$
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*Given an algorithm that solves Streett games in time $T(n, m, d)$, there is an algorithm that solves cost-Streett games in time $O(n \cdot T(2^d \cdot n, 2^d \cdot m, 2d))$.***
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## Overview of Results

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Cost-parity games with multiple cost functions (one for each odd color). Preliminary results:

- Complexity: between \textbf{PSPACE}-hard and \textbf{EXPTIME}
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Tackle stronger winning conditions:

- Max-automata: deterministic automata, with multiple counters than can be incremented and reset, acceptance condition is boolean combination of boundedness requirements
- Equivalent to WMSO$^+\text{U}$