Easy to Win, Hard to Master:
Optimal Strategies in Parity Games with Costs

Joint work with Martin Zimmermann

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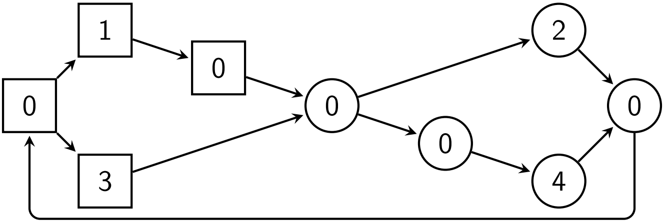
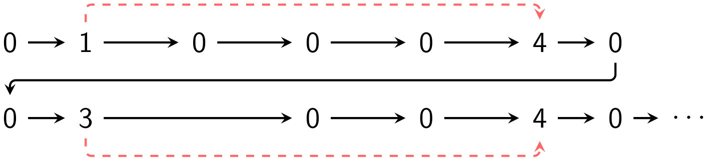
Parity Games

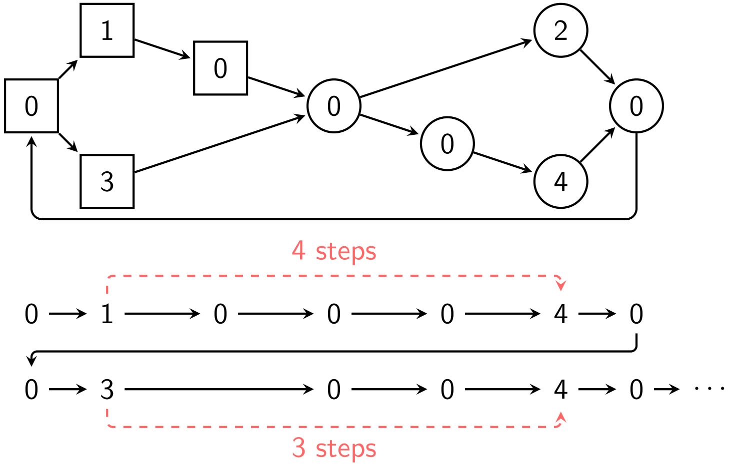




Deciding winner in $\mathrm{UP} \cap \text{co-}\mathrm{UP}$ Positional Strategies

Example due to (Fijalkow and Chatterjee, Infinite-state games, 2013)

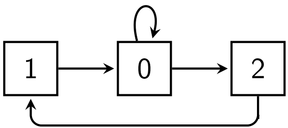
Finitary Parity / Parity Response Games

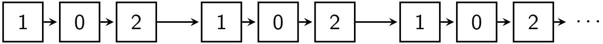


Goal for Player 0: Bound response times

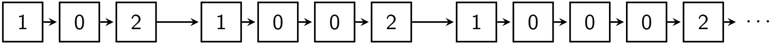
Example due to (Fijalkow and Chatterjee, Infinite-state games, 2013)

Another Example





Parity ✓ Finitary Parity ✓



Parity ✓ Finitary Parity ✗

- Player 1 wins from every vertex,
- but needs to stay longer and longer in vertex of color 0
  $\Rightarrow$ requires infinite memory

Decision Problem

Theorem (Chatterjee et al., Finitary Winning, 2009)

The following decision problem is in PTIME*:*

Input: *Finitary parity game* $\mathcal{G} = (\mathcal{A}, \mathrm{FinParity}(\Omega))$

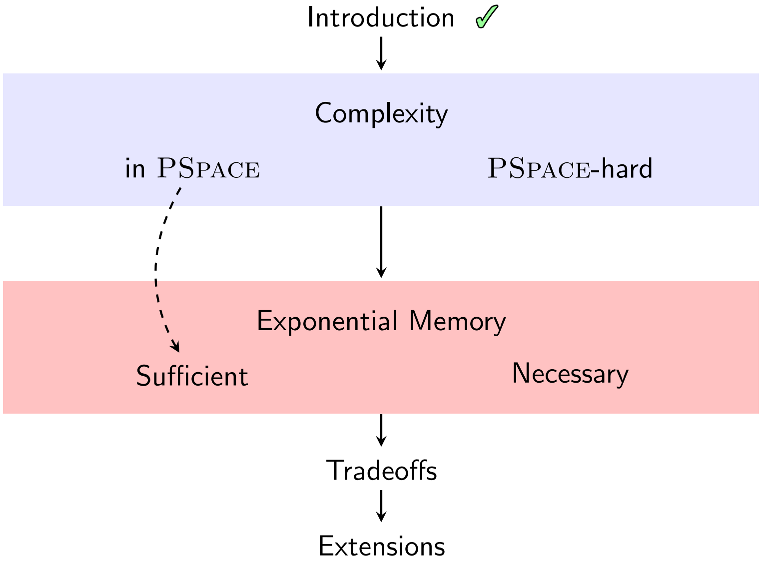
Question: *Does there exist a strategy* σ *with* $\mathrm{Cst}(\sigma) < \infty$*?*

Theorem

The following decision problem is PSPACE*-complete:*

Input: *Finitary parity game* $\mathcal{G} = (\mathcal{A}, \mathrm{FinParity}(\Omega))$*,*
bound $b \in \mathbb{N}$

Question: *Does there exist a strategy* σ *with* $\mathrm{Cst}(\sigma) \leq b$*?*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From Finitary Parity to Parity

Given: Finitary parity game $\mathcal{G} = (\mathcal{A}, \mathrm{FinParity}(\Omega))$, bound $b \in \mathbb{N}$.

Lemma

Deciding if Player 0 has strategy σ with $\mathrm{Cst}(\sigma) \leq b$ is in $\mathrm{PS{\small PACE}}$.

Idea: Simulate $\mathcal{G}$, keeping track of open requests explicitly.

Result: Parity game $\mathcal{G}'$ of exponential size.

Lemma

The winner of a play in $\mathcal{G}'$ can be decided after $p(|\mathcal{G}|)$ steps.

Algorithm:

Simulate all plays in $\mathcal{G}'$ on-the-fly for $p(|\mathcal{G}|)$ steps using an alternating Turing machine.

$\Rightarrow$ Problem is in $\mathrm{APT{\small IME}}$

(Chandra et al., Alternation, 1981)

$\Rightarrow$ Problem is in $\mathrm{PS{\small PACE}}$

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PSPACE-Hardness

Lemma

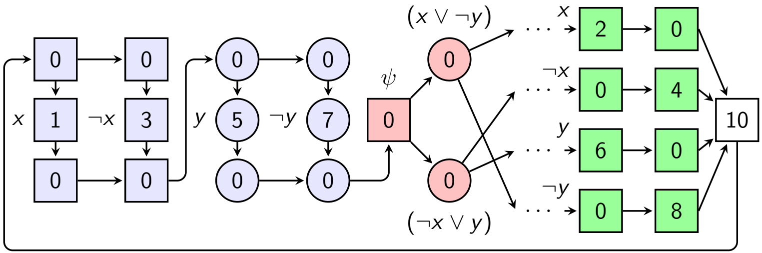
The following problem is $\text{PS}\textsc{pace}$*-hard: "Given a finitary parity game* $\mathcal{G}$ *and a bound* $b \in \mathbb{N}$*, does Player* 0 *have a strategy* σ *for* $\mathcal{G}$ *with* $\text{Cst}(\sigma) \leq b$*?"*

Proof

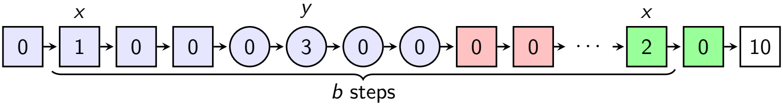
- By reduction from QBF
- Checking the truth of $\varphi = \forall x \exists y.\ (x \vee \neg y) \wedge (\neg x \vee y)$ as a two-player game (Player 0 wants to prove truth of φ):
  1. Player 1 picks truth value for x
  2. Player 0 picks truth value for y
  3. Player 1 picks clause C
  4. Player 0 picks literal ℓ from C
  5. Player 0 wins $\Leftrightarrow$ ℓ is picked to be satisfied in step 1 or 2

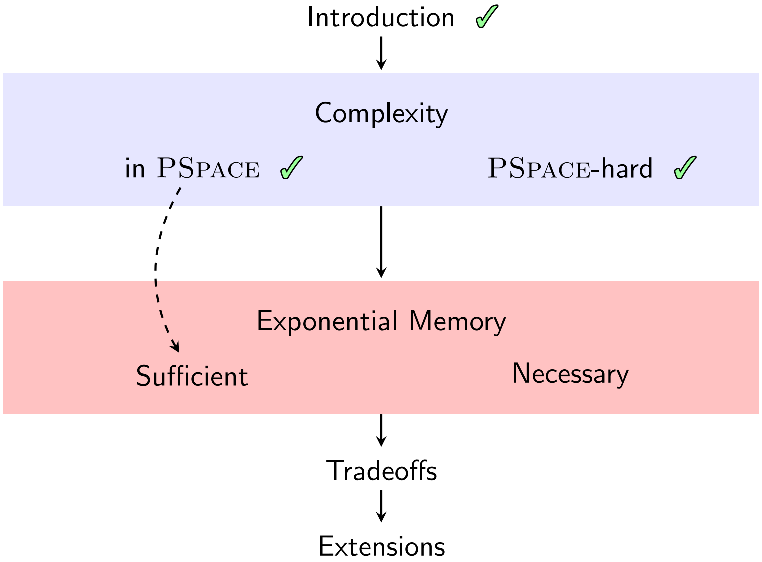
The Reduction

$$\varphi = \forall x \ \exists y \ . \ \overbrace{(x \lor \neg y) \land (\neg x \lor y)}^{\psi}$$



Choose bound b such that it enforces the following:

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Sufficient Memory (for Player 0)

Corollary

Let $\mathcal{G}$ be a parity game with costs with d odd colors.
If Player 0 has a strategy σ for $\mathcal{G}$ with $\mathrm{Cst}(\sigma) = b$, then she also has a strategy σ' with $\mathrm{Cst}(\sigma') = b$ and size $(b+2)^d = 2^{d \log(b+2)}$.

Follows from

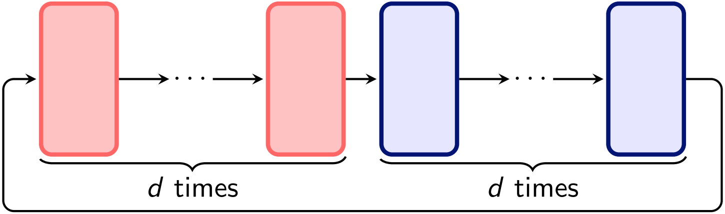
- proof of PSPACE-membership and
- positional strategies for parity games.

Memory Requirements (for Player 0)

Theorem

Optimal strategies for parity games require exponential memory.

Necessity: Construct family $\mathcal{G}_d$:



(Fijalkow and Chatterjee, Infinite-state games, 2013)

For optimal play:

Player 0 needs to store d choices of d possible values each

$\Rightarrow$ Player 0 requires $\approx 2^d$ many memory states

Memory Requirements (cont.)

Theorem

For every $d > 1$, there exists a finitary parity game $\mathcal{G}_d$ such that

- $|\mathcal{G}_d| \in \mathcal{O}(d^2)$ *and* $\mathcal{G}_d$ *has* d *odd colors, and*
- *every optimal strategy for Player* 0 *has at least size* $2^d - 2$.

Similar bounds (upper and lower) hold true for Player 1.

Corollary

Let $\mathcal{G}$ *be a parity game with costs with* d *odd colors.*
If Player 0 *has a strategy* σ *for* $\mathcal{G}$ *with* $\mathrm{Cst}(\sigma) = b$, *then she also has a strategy* σ' *with* $\mathrm{Cst}(\sigma') = b$ *and size* $(b + 2)^d = 2^{d \log(b+2)}$.

Introduction ✓

Complexity

in PSpace ✓ | PSpace-hard ✓

Exponential Memory

Sufficient ✓ | Necessary ✓

Tradeoffs

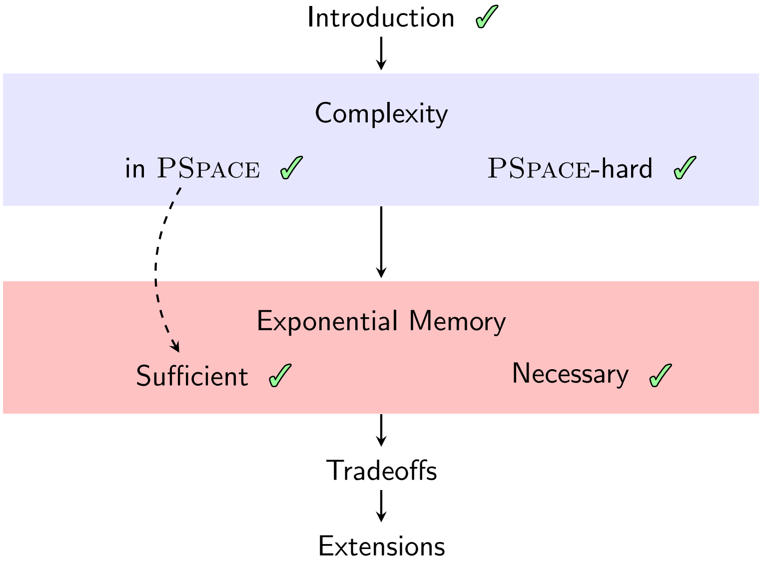
Extensions

Results so far

	Parity	Finitary Parity	
		Winning	Optimal
Complexity	UP ∩ co-UP	PTime	PSpace-comp.
Strategies	Pos.	Pos.	Exp.

Take-away: Forcing Player 0 to answer quickly in (finitary) parity games makes it harder

- to decide whether she can satisfy the bound
- for her to play the game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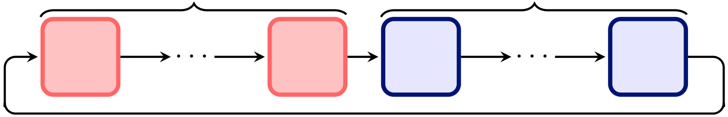
Tradeoffs

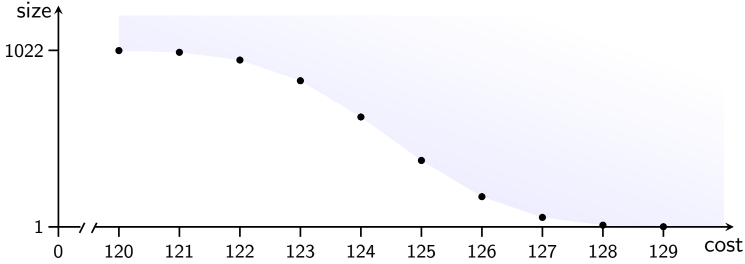


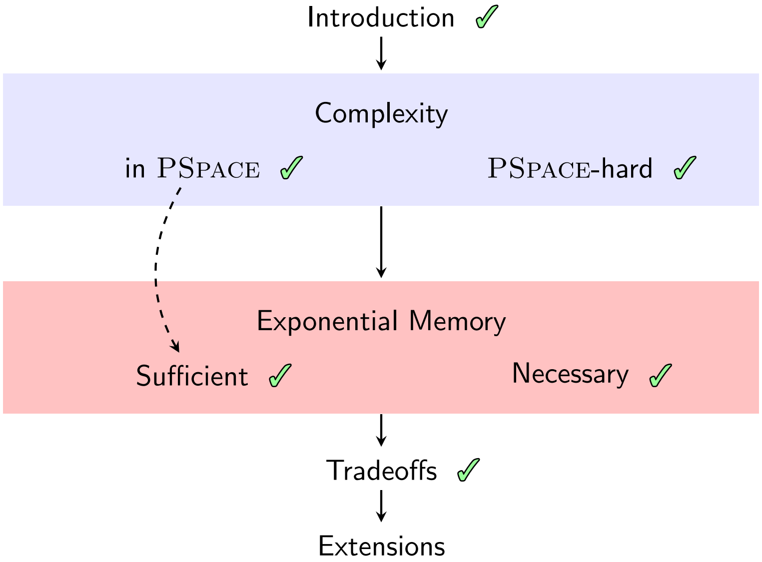
- **Recall:** Player 0 has winning strategy with cost $d^2 + 2d$ and size $2^d - 2$: store all d requests made by Player 1.
- **Smaller strategy:** Only store first i unique requests, then default to largest answer.
  $\Rightarrow$ achieves cost $d^2 + 3d - i$ and size $\sum_{j=1}^{i-1} \binom{n}{j}$
- These are the smallest strategies achieving this cost.

Tradeoffs

Theorem

Fix some finitary parity game $\mathcal{G}_d$ as before. For every i with $1 \leq i \leq d$ there exists a strategy σ_i for Player 0 in $\mathcal{G}_d$ such that σ_i has cost $d^2 + 3d - i$ and size $\sum_{j=1}^{i-1} \binom{d}{j}$.
Also, all these strategies are minimal for their respective cost.

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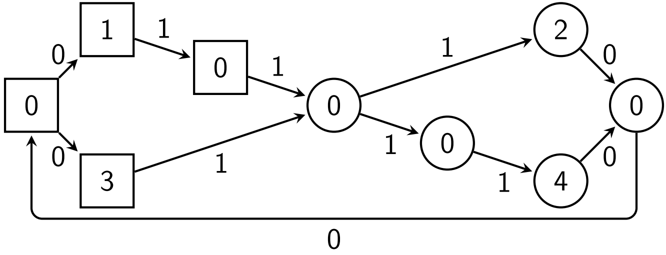
Extension 1: Parity Games with Costs



Finitary parity games are special case

⇒ PSpace-hard ⇒ Exp. memory necessary

Algorithm for finitary games works with some extensions

⇒ In PSpace ⇒ Exp. memory sufficient

Extension 2: Streett

Finitary Streett Games

- in parity game, large responses answer all lower requests
- in Streett games, there are requests and responses, but not hierarchical

Streett Games with Costs

- Streett condition and weights from $\{0, 1\}$

No jump in complexity:

- Solving finitary Streett games is already $\textsc{ExpTime}$-complete and exponential memory is necessary
  $\Rightarrow$ Appropriate $\mathcal{G}'$ can be solved directly

Streett Games with Costs

- Deciding winner $\textsc{ExpTime}$-complete
- Exponential memory necessary and sufficient

Conclusion

	Parity	Parity with Costs	
		Winning	Optimal
Complexity	$\mathrm{UP} \cap \mathrm{co}\text{-}\mathrm{UP}$	$\mathrm{UP} \cap \mathrm{co}\text{-}\mathrm{UP}$	PSpace-comp.
Strategies	Pos.	Pos.	Exp.

	Streett	Streett with Costs	
		Winning	Optimal
Complexity	co-NP	ExpTime	ExpTime-comp.
Strategies	Exp.	Exp.	Exp.

Slides available at `react.uni-saarland.de/people/weinert.html`